On Rank Dominance of Tie-Breaking Rules

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Abstract

Student assignment mechanisms use lotteries to ration seats in over-demanded schools. This paper compares two standard tie-breaking rules when paired with the student-proposing deferred acceptance algorithm. One uses a common lottery shared by all schools, and the other uses a separate, independent lottery at each school.

The model has a continuum of students and a finite set of schools with heterogeneous qualities, and students' preferences are generated from a multinomial-logit model induced by schools' qualities. A school is *popular* if the mass of students who rank it as their first choice exceeds its capacity. Assuming all schools are popular, it is shown that all students prefer a common lottery to separate lotteries, in the sense of first order stochastic dominance. This still holds for arbitrary distributions of students' preferences as long as the mass of students is sufficiently large. Moreover, under a common lottery, each school admits students who rank it more highly, in the sense of first order stochastic dominance. When non-popular schools exist, simulations confirm that under a common lottery every popular school still admits students who rank it higher.

1 Introduction

Lotteries are often used to allocate scarce resources without monetary transfers. How lotteries are conducted naturally affects distributional outcomes. This problem arises in student assignment mechanisms, when ties must be resolved in over-demanded schools (Erdil and Ergin, 2008a). There are two common types of lotteries in school choice mechanisms. One assigns each student a single random number, which is used by every school. The other assigns each student a different random number at each school. These lotteries, named single tie-breaking (STB) and multiple tie-breaking (MTB) rules, naturally result in different assignments. This paper uncovers distributional properties of students' ranks in stable student assignments under these lotteries when schools have heterogeneous qualities and students have random multinomial-logit-based preferences.

Previous empirical (Abdulkadiroğlu et al., 2009; De Haan et al., 2015) and theoretical (Arnosti, 2015; Ashlagi et al., 2015) studies find that a single lottery assigns more students to their top-ranked choices, but also more students to lower-ranked choices. Ashlagi and Nikzad (2016) further identify that in a random market with short supply, this trade-off vanishes and STB is in fact preferable to MTB in the sense of "approximate" first-order stochastic dominance. Their stylized model assumes that students' preference lists are generated independently and uniformly at random and every school has one seat.

This paper considers a general model for rationing seats using lotteries, where schools have heterogeneous qualities and capacities and students have a rich model of random preferences that takes into account schools' qualities. We note that in the case in which schools have identical qualities, preferences in our model are generated uniformly at random as in Ashlagi and Nikzad (2016). We are interested in comparing students' ranks distributions at each school (and not just in the entire market as done in previous studies).

The model we consider has n schools and a continuum of students, based on Azevedo and Leshno (2016); Abdulkadiroğlu et al. (2015). Each school j has a fixed quality $\mu_j > 0$ and capacity $q_j > 0$. Students have complete and strict preference orders over schools, drawn independently from a multinomial-logit (MNL) model induced by schools' qualities. That is, a student's top choice is school j with probability $\frac{\mu_j}{\sum_{k=1}^{n} \mu_k}$; more generally, the probability that school j is a student's most preferred school from a subset of schools $S \ni j$ equals $\frac{\mu_j}{\sum_{k \in S} \mu_k}$.[1] It is assumed (for simplicity and tractability) that schools are indifferent between

[1]Such preferences were also used in Kojima and Pathak (2009) and also referred to by popularity based preferences, e.g., in Gimbert et al. (2019). Moreover MNL preferences are often used to estimate demand in school choice problems (e.g., Shi (2015); Agarwal and Somaini (2018); Pathak and Shi (2013)).

all students, so priorities of students at each school are solely determined by lotteries. We compare student-optimal stable matching under the STB and MTB lotteries.

A key notion in the results is related to the demand for a school. A school is *popular* if the mass of students who rank it as their first choice exceeds its capacity. This notion generalizes the idea of over-demanded schools in Ashlagi and Nikzad (2016).[2]

The main results are established for the case in which all schools are popular. It is shown that every student prefers STB to MTB in the sense of first-order stochastic dominance (Theorem 1 Part 1). Moreover, every school admits students that rank it higher under STB than under MTB in a first-order stochastic dominance sense (Theorem 1 Part 2).

When the market also includes non-popular schools, then the first part of the theorem no longer holds (which is consistent with the empirical comparisons of MTB and STB). However, we conjecture that the second part still holds in every popular school (but not in other schools, as demonstrated by example). This conjecture is shown to hold in the case of three schools. Numerical simulations using data from New York City confirm our predictions despite natural deviations from our stylized model.

A rough intuition for the result is that when schools are sufficiently popular, a coordinated lottery essentially determines which students will be assigned, and among these students the allocation is efficient. A lottery at each school results in inefficiencies among assigned students who may wish to trade their assignments. For further intuition consider the following a simple example. Consider a market with two schools which have qualities $\mu_1 \geq \mu_2$. In the execution of the student-proposing deferred acceptance (DA) algorithm, at each round every unassigned student applies to her favorite choice to which she has not yet applied. Observe that, under STB, a student who is rejected from school 2 cannot be admitted to school 1 because her lottery number did not suffice to be admitted to a school in lower demand. This implies that, under STB, students assigned to the first school must rank it as their first choice. Moreover, under both lotteries, the same mass of students whose first choice is school 1 are rejected from that school after all students apply to their first choice. But more of these students will be admitted to their second choice under MTB than under STB, because under MTB they receive a new lottery number for school 2. DA will terminate after 2 rounds for STB, but for MTB the process will continue, with each round assigning more students to schools that they rank second. Observe that the outcome satisfies the results in Theorem 1.[3]

[2]With unit capacities and preferences drawn uniformly at random, every school is ranked as the first choice, on average, by more than one student if there are more students than seats.

[3]In fact, in this simple example, the result holds true even if the second school is not popular.

A direct analysis becomes challenging already with three schools and more subtle arguments are required that build on the cutoff characterization of stable matchings. Under a single lottery and MNL preferences the outcome has a simple structure and cutoffs have a closed form (Ashlagi and Shi, 2014). Under MTB, however, the cutoffs do not have a closed-form expression and we compare the two tie-breaking rules by developing bounds on the distribution of students' ranks.

This paper contributes to the analysis of rank distributions in matching markets where students' preferences are generated from a rich and empirically relevant model (Agarwal and Somaini, 2018). Techniques and intermediate results, which establish properties about the cutoff structures, may be of independent interest.

1.1 Related literature

This paper contributes to the literature on school choice. Abdulkadiroğlu and Sönmez (2003); Abdulkadiroğlu et al. (2009, 2005) apply matching theory to develop strategyproof mechanisms for school choice. Policy decisions surrounding school choice involve also resolving tie-breaking (Erdil and Ergin, 2008b; Abdulkadiroğlu et al., 2009), design of menus and priorities (Ashlagi and Shi, 2016; Dur et al., 2013; Shi, 2019) and diversity-related constraints (Ehlers et al., 2014; Echenique and Yenmez, 2015; Kominers and Sönmez, 2016).

Several papers analyze trade-offs between STB and MTB in addition to those discussed above. Arnosti (2019) compares single and independent lotteries in a model in which there is a continuum of schools, each of which has capacity for a finite number of students. The paper analyzes the effect of students having preference lists of varying length, and establishes a single crossing property between the cumulative rank distribution of students under STB and MTB (see also Ashlagi et al. (2015), which explains why STB assigns more students to their top choices in a model with random preferences).[4] Additionally, he shows that among students who submit short lists, the rank distribution under a single lottery stochastically dominates the corresponding distribution under independent lotteries. Our paper assumes a rich preference model over qualities but distinguishes between popular and non-popular schools to explain the source of these trade-offs.

Shi (2019) optimizes over the space of all priority-based mechanisms and finds that a single lottery maximizes the total utility of students when the utilities follow an MNL model.

[4] Abdulkadiroğlu et al. (2015) analyzes the cutoffs that clear the market in a continuum model and establish that STB is ordinally efficient (see also Che and Kojima (2010), Liu and Pycia (2012), and Ashlagi and Shi (2014)).

Our paper in contrast looks at the rank distributions under common and independent lotteries and identifies when these distributions exhibit a rank dominance relation.

Arnosti and Shi (2020) compare common and multiple lotteries in a dynamic model where agents have heterogeneous values for distinct items and heterogeneous outside options. They show that using independent lotteries for each item is equivalent to using a waitlist in which agents lose priority when they reject an offer, and that using a common lottery for each item improves the quality of match.

2 Model

We study a large matching model based on the framework in Azevedo and Leshno (2016). There is a finite set of schools $S = \{1, ..., n\}$ and a continuum of students with mass N. Each school $j \in S$ has capacity $q_j > 0$. Each student has a strict preference ranking, which is a linear order over all schools. Let Π_n be the set of all permutations of n elements. A *school choice problem* is given by $C = (m, q, N)$ where m is a probability measure over Π_n and $q = (q_1, ..., q_n)$ is the vector of schools' capacities.

Tie breakers. We assume that schools are indifferent between all students, so priorities of students at each school are solely determined by lotteries. Each student i is assigned a vector of lottery numbers $L^i \in [0,1]^n$, where L^i_j is student i's lottery number at school j. Each school j is assumed to prefer students in decreasing order of their lottery number at j. To generate lottery numbers for students, the following definition will be helpful.

Definition 1. *A tie-breaking rule is a probability measure ν defined on $[0,1]^n$ where each marginal of ν is non-atomic.*[5]

Requiring that the marginals of a tie-breaking rule are non-atomic ensures that each school has strict preferences over students. A *school economy* is given by $E = (C, \nu)$, where C is a school choice problem and ν is a tie-breaking rule.

Two commonly applied tie-breaking rules are studied, Single Tie-Breaking (STB) and Multiple Tie-Breaking (MTB).Under STB each student receives the same lottery number for all schools, uniformly distributed on $[0,1]$. So, STB is the uniform measure on the line $\{(x, x, ..., x) : x \in [0,1]\}$. Under MTB each student receives a lottery number independently for each school, where each number is chosen uniformly on $[0,1]$. So, MTB is the uniform measure on $[0,1]^n$.

[5] Where ν is defined on the Lebesgue σ-algebra on $[0,1]^n$.

For tractability we do not allow students to have differing priorities before tie-breaking, as is common in actual school choice problems, but we use these priorities in simulations with real data in Section 5.

Matching, stability and cutoffs. Consider a school economy $E = (m, q, N, \nu)$ with tie-breaking ν. Let $T = \Pi_n \times [0,1]^n$ be the set of all pairs of student preferences and lottery numbers. A *matching* is a function $f : S \cup T \to 2^T \cup S \cup \{\emptyset\}$ such that

1. For all $i \in T, f(i) \in S \cup \{\emptyset\}$.
2. For all $j \in S$, $f(j) \subseteq T$ is $(m \times \nu)$-measurable and $(m \times \nu)(f(j)) \leq q_j$, where $(m \times \nu)$ is the product measure between m and ν.
3. For all $i \in T$ and $j \in S$, $j = f(i)$ if and only if $i \in f(j)$.
4. For any sequence $i^k = (P^k, L^k) \in T$ and $i = (P, L) \in T$, with L^k converging to L and weakly decreasing with k in each component, there is some K such that $f(i^k) = f(s)$ for all $k > K$.

The first condition ensures that a student is assigned either to a school (and thus matched) or to the empty set (remaining unmatched). The second condition ensures that the mass of students assigned to each school does not exceed its capacity. The third condition ensures that if a student is assigned to a school, the school is matched to the student. The fourth technical condition eliminates multiplicities of matchings that differ by a set of measure 0 (Azevedo and Leshno, 2016).

A matching f is *stable* if there is no student i and school j such that i strictly prefers j to $f(i)$, and either there is some $i' \in f(j)$ such that $L_j^s > L_j^{i'}$ or j has excess capacity. Azevedo and Leshno (2016) show that every stable matching corresponds to a set of cutoffs $c = (c_1, ..., c_n) \in [0,1]^n$, where every student i is matched to her most preferred school j for which her lottery number exceeds the cutoff ($L_j^i \geq c_j$).

Denote by f^ν the (student-optimal) stable matching for the school economy. So f^{STB} and f^{MTB} denote that student-optimal matching when the tie-breaking rules are STB and MTB, respectively.[6] When it is clear from the context we denote by $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ the cutoffs under the matchings f^{STB} and f^{MTB}, respectively.

The STB cutoffs can be calculated in closed form (Ashlagi and Shi, 2014). This is not the case for the MTB cutoffs, but these can be computed through an iterative algorithm, which progressively increases the cutoffs to clear the market, and converges to β.

[6] Azevedo and Leshno (2016) show that under regularity conditions the stable matching is unique.

Ranks and dominance. For a given matching, the *rank* of a student is the position on the student's preference list of the school to which the student is assigned. For example, if a student is matched to her second choice, then her rank is two.

Consider a school choice problem and a tie-breaking rule $\nu \in \{STB, MTB\}$. Denote by R_j^ν the distribution of student ranks at school j in the stable matching f^ν. That is, R^ν is the n-dimensional vector in which its k-th element is the fraction of students assigned to school j with rank k in f^ν.

For a preference order $P \in \Pi_n$ let R_P^ν denote the distribution of ranks of students with preference P who are matched in f^ν. That is, R_P^ν is the $n-$dimensional vector in which its k-th element is the fraction of students with preference order P who are assigned their k-th ranked school in f^ν.

Observe that for any $j \in S$ and $P \in \Pi_n$ the vectors R_j^ν and R_P^ν are *stochastic vectors*, i.e., vectors with nonnegative entries that sum to 1.

Definition 2. *If $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$ are stochastic vectors with equal length n, v rank-dominates w, indicated by $v \succeq w$ if for all $k \in \{1, ..., n\}$,*

$$\sum_{j=1}^{k} v_j \geq \sum_{j=1}^{k} w_j.$$

Observe that rank dominance is equivalent to first-order stochastic dominance but with the order of the two vectors reversed. This definition is adopted for clarity of exposition, since in this setting lower student ranks are preferable to higher ranks in terms of welfare.

Definition 3. STB dominates MTB for students with preference order $P \in \Pi_n$ *if*

$$R_P^{STB} \succeq R_P^{MTB}.$$

Definition 4. STB dominates MTB at school j *if*

$$R_j^{STB} \succeq R_j^{MTB}.$$

3 Main Results

The main results are established in a setting in which students' preferences are drawn from a multinomial-logit (MNL) preference model (see Section 6 for extensions). Informally, each school j has quality $\mu_j > 0$. Then a student's first choice (most preferred school) is drawn independently from the multinomial distribution with a weight for each school that equals its quality. The student's second choice is then drawn similarly from all remaining schools, and so forth.

Definition 5. *A school choice problem* $C = (m, q, N)$ *has* MNL preferences *with school qualities* $\mu_1, ..., \mu_n$ *if for any preference order* $P = (P_1, ..., P_n) \in \Pi_n$,

$$m(P) = \prod_{k=1}^{n} \frac{\mu_{P_k}}{(\sum_{p=1}^{n} \mu_p) - \sum_{p=1}^{k-1} \mu_{P_p}}.$$

When clear from the context, we sometimes say that students have MNL preferences.

We note that this definition is just multiplying the chance that P_1 is the student's first choice by the chance that P_2 is the student's second choice, and so on. Throughout the paper, when a school choice problem has MNL preferences, we assumed that schools are indexed such that

$$\frac{\mu_1}{q_1} \geq ... \geq \frac{\mu_n}{q_n}.$$

Definition 6. *A school j is* popular *if the mass of students who rank it as their first choice is at least the capacity of j.*

Note that under MNL preferences, school $j \in S$ is popular if and only if $N\mu_j \geq q_j$.

Theorem 1. *Consider a school choice problem that has MNL preferences and every school is popular. Then*

1. *STB dominates MTB for students with any preferences.*

2. *STB dominates MTB at every school.*

While Theorem 1 assumes that all schools are popular, we believe that STB dominates MTB at every popular school, even when some schools are non-popular (see simulations in Section 5). We prove this in the case when $n = 3$.

Proposition 1. *Consider a school choice problem with three schools and MNL preferences. Then STB dominates MTB at every popular school.*

Conjecture 1. *Consider a school choice problem with MNL preferences. Then STB dominates MTB at every popular school.*

The following example illustrates Theorem 1.

Example 1. *Consider a school choice problem with two schools and MNL preferences. The schools have capacities $q = (\frac{1}{2}, \frac{1}{2})$ and qualities $\mu = (3, 1)$, and the mass of students is $N = 2$. So, a mass of $\frac{3}{2}$ students have preferences $(1, 2)$, and a mass of $\frac{1}{2}$ students have preferences $(2, 1)$. It can be computed (Proposition 2) that the STB cutoffs are $\alpha = (\frac{2}{3}, \frac{1}{2})$. Then, students with preferences $(1, 2)$ have a $\frac{1}{3}$ chance of being assigned to school 1 (if their lottery number is at least $\frac{2}{3}$), and a $\frac{1}{6}$ chance of being assigned to school 2 (if their lottery number is between $\frac{1}{2}$ and $\frac{2}{3}$). Similar reasoning follows for students with preferences $(2, 1)$. The rank distributions of the schools are then*

$$R_1^{STB} = (1, 0), \;\; R_2^{STB} = (\frac{1}{2}, \frac{1}{2})$$

and the rank distributions of the students are

$$R_{(1,2)}^{STB} = (0.67, 0.33), \;\; R_{(2,1)}^{STB} = (1, 0).$$

The MTB cutoffs can be computed by an iterative process to be $\beta = (0.73, 0.69)$. Each student with preference (P_1, P_2) has a $(1 - \beta_{P_1})$ chance of being assigned to school P_1, and a $\beta_{P_1}(1 - \beta_{P_2})$ chance of being assigned to school P_2. The rank distributions are then

$$R_1^{MTB} = (0.84, 0.16), \;\; R_2^{MTB} = (0.31, 0.69)$$

and

$$R_{(1,2)}^{MTB} = (0.56, 0.44), \;\; R_{(2,1)}^{MTB} = (0.68, 0.32).$$

Observe that

$$R_{(1,2)}^{STB} \succeq R_{(1,2)}^{MTB}, \;\; R_{(2,1)}^{STB} \succeq R_{(2,1)}^{MTB}$$

and

$$R_1^{STB} \succeq R_1^{MTB}, \;\; R_2^{STB} \succeq R_2^{MTB}.$$

Note that $\alpha_2 = \beta_1 \beta_2$, because the probability of a student being unassigned is equal under STB and MTB.

The intuition for the dominance of STB over MTB when there are only two schools is relatively simple. Under STB, students who first apply to school 2 but are rejected remain unassigned, whereas under MTB these students have a chance to apply to school 1 and displace students who otherwise would have been assigned there. So students are more likely to receive their second choice school under MTB, and thus are less likely to receive their first choice. In fact, Theorem 1 applies when there are only two schools even if the schools are not popular.

When there are at least three schools, this intuition breaks down, and even proving the theorem for when there are three schools is nontrivial. We were unable to find a simple or inductive proof of the theorem, as adding an additional school to a school choice problem has a complicated effect on schools and students of different types.

In Appendix B we demonstrate the necessity of the conditions in Theorem 1 through Example 2, which shows that STB may not dominate MTB at schools that are non-popular, even if these schools are over-demanded (loosely speaking, a school is over-demanded if the mass of students applying to the school is larger than its capacity). In the example, MTB dominates STB at school 3, and school 3 is an over-demanded but non-popular school. In general it is possible for neither STB nor MTB to dominate the other at a non-popular school.

4 Analysis

4.1 Preliminary Results

This section provides properties that will be useful in our proofs. Consider a school choice problem $C = (m, q, N)$ that has MNL preferences with school qualities $\mu_1, \ldots, \mu_n$. The next result shows that students' (MNL) preferences can be generated equivalently using a stochastic process involving exponential clocks.

Claim 1. *Consider drawing a student's preferences by the following process. For each school j, let X_j be an independent exponential random variable with rate μ_j. For each $k \in [n]$, let $X^{(k)}$ be the kth smallest value in $X_1, ..., X_n$. For each school j, if $X_j = X^{(k)}$ then set school j as the student's kth rank. The distribution of preferences generated by this process is equivalent to the distribution of preferences generated by the MNL preference model.*

The above process can be interpreted as n exponential clocks, where X_j is the time that clock j rings, and the student ranks the schools in the order of the time the clocks ring. We call this method of drawing student preferences the clock process.

Proof. For a student i and school j, the probability that i ranks school j as her first choice in the clock process is

$$P\{X_j = X^{(1)}\} = P\{X_j = \min(X_1, ..., X_n)\} = \frac{\mu_j}{\sum_{p=1}^{n} \mu_p}.$$

Thus, the distribution of i's first choice is the same in both the clock process and the MNL preference model. Now suppose that in the clock process, clock k is the first clock to ring and $X_k = t$. Conditional on this event, by the memoryless property of exponential random variables, for each school $k' \neq k$ we have that $X_{k'} - t$ is exponentially distributed with rate $\mu_{k'}$. So, the probability that school $k' \neq k$ is the next clock to ring is

$$\frac{\mu_{k'}}{(\sum_{p=1}^{n} \mu_p) - \mu_k},$$

which is the same as in the MNL preference model. Continuing this reasoning inductively proves the claim. □

Next we provide properties of the STB and MTB cutoffs.

Proposition 2 (Ashlagi and Shi (2014))**.** *The STB cutoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$ are the following:*

$$\alpha_k = 1 - \min\{\sum_{j=1}^{k-1} \frac{q_j}{N} + \frac{r_k q_k}{N\mu_k}, 1\},\ k = 1, ..., n, \tag{1}$$

where $r_k = \sum_{j=k}^{n} \mu_j$.

Note that the α_k values are decreasing in k.

Proposition 3. *Suppose all schools are popular. Then for all $k \in [n]$ the MTB cutoffs $\beta = (\beta_1, \ldots, \beta_n)$ satisfy*

$$\prod_{j=1}^{k} \beta_j \geq 1 - \frac{\sum_{j=1}^{k} q_j}{N} \cdot \frac{\sum_{j=1}^{n} \mu_j}{\sum_{j=1}^{k} \mu_j}.$$

Proof. Fix a randomly chosen student i. For a subset of schools $G \subseteq S$, let Z_G be the event that i is not assigned to a school in G. Let $L = [k]$.

Since $Z_L \cap Z_{S \setminus L}$ is the event that i is unassigned, $P(Z_L \cap Z_{S\setminus L}) = \prod_{j=1}^{n} \beta_j$. Moreover, i will not be assigned to a school in $S \setminus L$ if her lottery number at each of these schools does not exceed the cutoff. Therefore $P(Z_{S\setminus L}) \geq \prod_{j=k+1}^{n} \beta_j$. By Bayes' rule,

$$P(Z_L | Z_{S\setminus L}) \leq \prod_{j=1}^{k} \beta_j. \tag{2}$$

Let M be the mass of students that are not assigned to schools in $S \backslash L$. Since the total capacity of the schools in L is $\sum_{j=1}^{k} q_j$,

$$M = N - \sum_{j=k+1}^{n} q_j \quad \text{and} \quad M(1 - P(Z_L | Z_{S \backslash L})) = \sum_{j=1}^{k} q_j. \tag{3}$$

Since all schools are popular, $N\mu_j \geq q_j$ for all $j \in [n]$, implying that

$$\sum_{j=k+1}^{n} q_j \leq N \sum_{j=k+1}^{n} \mu_j. \tag{4}$$

By (2), (3) and (4) we obtain that

$$\prod_{j=1}^{k} \beta_j \geq 1 - \frac{\sum_{j=1}^{k} q_j}{M} = 1 - \frac{\sum_{j=1}^{k} q_j}{N - \sum_{j=k+1}^{n} q_j} \geq 1 - \frac{\sum_{j=1}^{k} q_j}{N(1 - \sum_{j=k+1}^{n} \mu_j)} = 1 - \frac{\sum_{j=1}^{k} q_j}{N \sum_{k=1}^{k} \mu_j}.$$

□

The next property is a simple observation about stochastic vectors. For convenience, define the notation $[n] = \{1, ..., n\}$.

Definition 7. *Let v and w be stochastic vectors of length n. Vector v* crosses under *w if there is some $k \in [n]$ such that $v(p) \leq w(p)$ when $1 \leq p \leq k$, and $v(p) \geq w(p)$ when $k < p \leq n$.*

Claim 2. *Let v and w be stochastic vectors and suppose v crosses under w. Then $w \succeq v$.*

Proof. Suppose that v crosses under w, and $k \in [n]$ satisfies $v(p) \leq w(p)$ for all $1 \leq p \leq k$ and $v(p) \geq w(p)$ for all $k < p \leq n$. If $t \leq k$, then

$$\sum_{p=1}^{t} v(p) \geq \sum_{p=1}^{t} w(p).$$

If $t \geq k$, then

$$\sum_{p=1}^{t} v(p) = 1 - \sum_{p=t+1}^{n} v(p) \leq 1 - \sum_{p=t+1}^{n} w(p) = \sum_{p=1}^{t} w(p).$$

□

4.2 Proof of Theorem 1 Part One

Proof. Consider a school choice problem $C = (m, q, N)$ with n schools, satisfying MNL preferences, with school qualities μ. Without loss of generality assume $\sum_{j=1}^{n} \mu_j = 1$. Fix a student i with preferences $P = (P_1, ...P_n)$. Let $(R_P^{STB})_{\leq k}$ denote the probability that i will be assigned to one of her top k choices under STB, and let $(R_P^{MTB})_{\leq k}$ denote the same under MTB. Then $R_P^{STB} \succeq R_P^{MTB}$ if and only if for all $k \in [n]$

$$(R_P^{STB})_{\leq k} \geq (R_P^{MTB})_{\leq k}.$$

Fix an arbitrary integer k, where $1 \leq k \leq n$, and let $m_k = \max\{P_1, ..., P_k\}$. Since the STB cutoffs are weakly decreasing in the index (of schools), i will be assigned to a school in $\{P_1, ..., P_k\}$ if and only if she has lottery number at least α_{m_k}, so

$$(R_P^{STB})_{\leq k} = 1 - \alpha_{m_k}.$$

Under MTB, i will not be assigned to a school in $\{P_1, ..., P_k\}$ if and only if for each school $j \in \{P_1, ..., P_k\}$ her lottery number for j is below β_j. So

$$(R_P^{MTB})_{\leq k} = 1 - \prod_{j \in \{P_1, ..., P_k\}} \beta_j.$$

Since k is chosen arbitrarily, it is sufficient to show that

$$\prod_{j \in \{P_1, ..., P_k\}} \beta_j \geq \alpha_{m_k}. \tag{5}$$

Since $m_k \geq k$, then $\alpha_{m_k} \leq \alpha_k$. Therefore it is sufficient to show that

$$\prod_{j=1}^{k} \beta_j \geq \alpha_k. \tag{6}$$

This will be done by comparing the cutoffs for the school choice problem C to the cutoffs for another school choice problem C', which is similar to C but contains additional schools and a larger mass of students.

Let $C' = (m', q', N')$ be a school choice problem with $n' > n$ schools, where students have MNL preferences. Let $\mu'_1, ..., \mu'_n$ be the school qualities in C'. For $j \in [n]$, let $q'_j = q_j$ and

$\mu'_j = \mu_j$, and assume that

$$\frac{\mu_n}{q_n} \geq \frac{\mu'_{n+1}}{q'_{n+1}} \geq ... \geq \frac{\mu'_{n'}}{q'_{n'}}.$$

Let $N' = N\sum_{j=1}^{n'} \mu'_j$. Note that for each school $j \in [n]$, the mass of students who rank j their top choice in C is equal to the mass of students who rank j their top choice in C'. Let $\alpha' = (\alpha'_1, ..., \alpha'_{n'})$ and $\beta' = (\beta'_1, ..., \beta'_{n'})$ be the STB and MTB cutoffs for C', respectively. For each $j \in [n]$, let

$$\gamma_j = \prod_{p=1}^{j} \beta_p$$

and for each $j \in [n']$, let

$$\gamma'_j = \prod_{p=1}^{j} \beta'_p.$$

Then we must show that $\gamma_k \geq \alpha_k$. Note that since α_n and γ_n are the probabilities that a student will be assigned to any school under STB and MTB, respectively, we have

$$\alpha_n = \gamma_n = 1 - \frac{\sum_{j=1}^{n} q_j}{N}. \tag{7}$$

So, assume $k < n$. In the remainder of the proof, we present two lemmas, show how the lemmas imply the theorem, and finally prove the lemmas.

Lemma 1. *For all $j \in [n]$, $\alpha'_j \leq \alpha_j$.*

Lemma 2. *$\gamma'_n \geq \gamma_n$.*

We apply the two lemmas to complete the proof. The lemmas essentially say that when schools are added to a school choice problem and the mass of students is increased accordingly, the STB cutoffs decrease while the MTB cutoffs increase. We will use this fact in reverse: when schools are removed and the mass of students decreased accordingly, the STB cutoffs increase while the MTB cutoffs decrease.

Consider the school choice problem $C'' = (m'', q'', N'')$ satisfying MNL preferences, containing $k < n$ schools with qualities $\mu_1, ..., \mu_k$. For all $j \in [k]$, let $q''_j = q_j$, and let

$$N'' = N\sum_{j=1}^{k} \mu_j.$$

We know $\alpha''_k = \gamma''_k$ since C'' has k schools; thus, Lemmas 1 and 2 imply that $\gamma_k \geq \alpha_k$, which completes the proof.

In the remainder of this section we prove the two lemmas.

Proof of Lemma 1: For a given school choice problem with n schools satisfying MNL preferences, consider computing the α values by the following "water-filling algorithm." Let each school j be able to hold a mass q_j of water, and a total mass N of water needs to be poured. The algorithm starts at time 0 and in stage 1. At time 0, all schools are empty. During stage 1, water is poured into each school j at a rate of $\frac{N\mu_j}{\mu_1+...+\mu_n}$. Stage 1 concludes when school 1 is filled to capacity. Then the next stage begins. During stage k, schools $k-1$ have already been filled to capacity, and the water poured into them "spills over" into schools $k, ..., n$: during stage k each school $j \in \{k, ..., n\}$ fills at a rate of $\frac{N\mu_j}{\mu_k+...+\mu_n}$. It can be shown that for each school j, $1-\alpha_j$ is the time that j becomes full (that is, the time that stage $j+1$ begins).

Now, consider using the water-filling algorithm on C and C', to compute α and α'. We prove the lemma inductively over k, showing that $\alpha'_k \leq \alpha_k$ for all $1 \leq k \leq n$. As a base case, in both problems school 1 fills at a rate of $N\mu_1$, so $\alpha'_1 = \alpha_1$. Now for the inductive step, assume for some $k < n$, $\alpha'_k \leq \alpha_k$. Note that for both problems, before time $1-\alpha_k$ the ratio of the rate that school $k+1$ fills to the rate that school k fills is

$$\frac{\mu_{k+1}}{\mu_k + \mu_{k+1}}.$$

Furthermore, at the end of stage k, school j has been filled with mass q_k. Thus, in both problems, at the end of stage k school $k+1$ has been filled with mass

$$\frac{q_k\mu_{k+1}}{\mu_k + \mu_{k+1}}.$$

For C, stage k concludes at time $1-\alpha_k$, and in stage $k+1$ school $k+1$ fills at rate

$$r = N\sum_{j=1}^{k}\mu_j\frac{\mu_{k+1}}{\mu_{k+1}+...+\mu_n} + N\mu_{k+1}.$$

In C', stage k concludes at time $1-\alpha'_k$, and then in stage $k+1$ school $k+1$ fills at rate

$$r' = N(\mu_1 + ... + \mu_{n'})\sum_{j=1}^{k}\mu_j\frac{\mu_{k+1}}{\mu_{k+1}+...+\mu_{n'}} + N\mu_{k+1}.$$

Observe that $r' < r$ since

$$\frac{\mu_1 + \ldots + \mu_{n'}}{\mu_{k+1} + \ldots + \mu_{n'}} \leq \frac{1}{\mu_{k+1} + \ldots + \mu_n},$$

which follows from the following inequality: if $a, b, c > 0$ and $b \leq a$, then

$$\frac{a+c}{b+c} \leq \frac{a}{b}.$$

Now by assumption $\alpha'_j \leq \alpha_j$, so for C' stage $k+1$ begins at a later time than stage $k+1$ begins for C. Furthermore, since $r \geq r'$, during stage $k+1$ school $k+1$ fills at a slower rate for C' than for C. Thus, $1 - \alpha'_{k+1} \geq 1 - \alpha_{k+1}$, so $\alpha'_{k+1} \leq \alpha_{k+1}$. This concludes the induction and the proof of the lemma. □

Proof of Lemma 2: Observe that α_n is the probability that a student is not assigned to any school under STB, and γ_n is the same under MTB. Thus,

$$\gamma_n = \alpha_n = 1 - \frac{\sum_{j=1}^{n} q_j}{N}.$$

Proposition 3 gives

$$\gamma'_n \geq 1 - \frac{\sum_{j=1}^{n} q'_j}{N'} \cdot \frac{\sum_{j=1}^{n'} \mu'_j}{\sum_{j=1}^{n} \mu'_j} = 1 - \frac{\sum_{j=1}^{n} q_j}{N} = \gamma_n,$$

where the last equality follows from Equation 7. □

□

4.3 Proof of Theorem 1 Part Two

The proof uses an auxiliary student assignment process, referred to by virtual MTB (VMTB), which assigns (or leaves unassigned) *each* student independently as follows.

VMTB independent assignment process

> Input: vector of cutoffs $\beta' = (\beta'_1, \beta'_2, \ldots, \beta'_n)$. Initialize: $k = 1$.
>
> Step k: Let j be the school that is the student's kth rank. The student applies to school j. With probability $1 - \beta'_j$, the school admits the student and the process ends. Otherwise, the student is rejected from j. If $k = n$, the student remains unassigned and the process terminates. Otherwise, increase k by one, and go to the next step.

We refer to the VMTB assignment process with inputs β' simply as VMTB(β'). Note that the VMTB process may violate schools' capacities. However, due to a result by Azevedo and Leshno (2016), the process generates the MTB assignment with the "correct" input:[7] Observe that by construction, if β are the MTB cutoffs, then the assignment under VMTB(β) is equivalent to the assignment under MTB.

We fix notation for the distribution of student ranks under VMTB. Let β' be a vector of cutoffs, and $C = (m, q, N)$ be a school choice problem. Let $q^{\beta'} = (q_1^{\beta'}, ..., q_n^{\beta'})$, where $q_j^{\beta'}$ is the mass of students assigned to school j under VMTB(β'). For each school $j \in S$, let $R_j^{\beta'}$ denote the value of R_j^{MTB} for the school choice problem $C^{\beta'} = (m, q^{\beta'}, N)$. That is, $R_j^{\beta'}$ is the value of R_j^{MTB} when the students are assigned according to VMTB(β'). So, $R_j^{\beta} = R_j^{MTB}$ for every school j.

Consider a school choice problem $C = (m, q, N)$ with n schools, satisfying MNL preferences, with school qualities μ. Without loss of generality assume $\sum_{k=1}^{n} \mu_k = 1$. Fix a school j; we will show that $R_j^{STB} \succeq R_j^{MTB}$. Let β be the market-clearing cutoffs for C under MTB, let $\beta^0 = (\beta_1, ..., \beta_j, 0, ..., 0)$, and let $V_j = R_j^{\beta^0}$. The proof of the theorem proceeds by first showing that $R_j^{STB} \succeq V_j$ in Lemma 3, and then showing that $V_j \succeq R_j^{MTB}$ in Lemma 4. By the transitivity of the rank dominance relation, it will follow that $R_j^{STB} \succeq R_j^{MTB}$. Here we briefly sketch the proofs of the two lemmas, and leave the full proofs for the Appendix.

Lemma 3. $R_j^{STB} \succeq V_j$.

Proof sketch for Lemma 3. The proof of the lemma proceeds by constructing a stochastic vector W of length n, and then showing that both $R_j^{STB} \succeq W$ and $W \succeq V_j$. These relations are shown by proving that W crosses under R_j^{STB} and that V_j crosses under W as in Definition 7, and then applying Claim 2.

Observe that under both VMTB(β^0) and STB, a student can only be assigned to school j if she prefers j to all schools $j' > j$. Let A_j denote the set of students who prefer j to all schools $j' > j$. We construct W by setting $W(1) = R_j^{STB}(1)$, and for $k \geq 2$ we set $W(k)$ to be proportionate to the mass of students who rank school j their kth choice and are in A_j. Note that for $k > j$, this results in $W(k) = 0$.

We show that W crosses under R_j^{STB} by proving that for $k \geq 2$, conditional on a student being in A_j and ranking j her kth choice, the probability that the student is assigned to j is weakly decreasing in k. But $W(1) = R_j^{STB}(1)$, and for $k \geq 2$ the $W(k)$ values were chosen as if conditional on a student being in A_j and ranking j her kth choice, the probability she

[7] Azevedo and Leshno (2016) show that a stable matching corresponds to a set of cutoffs where each student is assigned to her most preferred school, in which her lottery number exceeds the cutoff.

is assigned to j is equal over k. Since W and R_j^{STB} are normalized to have the same sum, it is straightforward to show that then W crosses under R_j^{STB}.

To complete the proof of the lemma, we show that V_j crosses under W by proving that $W(1) \geq V_j(1)$, proving a lower bound on $V_j(k)$ for $k \geq 2$, and finally showing that $W(k)$ does not exceed this lower bound for $k \geq 2$. $\square$

To complete the proof of the theorem, it remains to show that $V_j \succeq R_j^{MTB}$. Recall that β are the MTB cutoffs, and $\beta^0 = (\beta_1, ..., \beta_j, 0, ..., 0)$. Recall that we have $R_j^{MTB} = R_j^{\beta}$, and we defined $V_j = R_j^{\beta^0}$. Since $\beta_k \geq \beta_k^0$ for all $k \in [n]$, the following lemma implies that $V_j \succeq R_j^{MTB}$.

Lemma 4. *Let β and β' be vectors of cutoffs such that $\beta_k \leq \beta'_k$ for all $k \in [n]$. Then $R_j^{\beta} \succeq R_j^{\beta'}$.*

Proof sketch for Lemma 4. To prove the lemma we fix a school $j' \neq j$ and show that $R_j^{\beta} \succeq R_j^{\beta'}$ when $\beta'_k = \beta_k$ for all $k \neq j'$. This would prove the claim because this special case can be applied successively. Let β and β' be sets of cutoffs such that $\beta_k = \beta'_k$ for all $k \neq j'$, and $\beta_{j'} \leq \beta'_{j'}$. The major step in the proof is to show that $R_j^{\beta'} \succeq R_j^{\beta}$ when $\beta_{j'} = 0$ and $\beta'_{j'} = 1$. Intuitively, if the cutoff for j' is reduced from 1 to 0, then the mass of students who would be assigned to j who prefer j' will now be assigned to j' instead. We show that this mass of students is rank-dominated by the rest of the students who are assigned to j, so reducing the cutoff for j' improves the rank distribution of students at j. Finally, we show how this fact can be used to prove $R_j^{\beta'} \succeq R_j^{\beta}$ for general values of $\beta'_{j'}$, by interpolating between $R_j^{\beta'}$ when $\beta_{j'} = 0$, and $R_j^{\beta'}$ when $\beta_{j'} = 1$. $\square$

5 Experiments

We next run a numerical experiment using data from a large US school district for the 2018 kindergarten assignment, in which over four thousand students were assigned to over 70 different schools. This experiment provides some evidence for the conjecture that STB dominates MTB at all popular schools, even if non-popular schools are present. There are two main deviations from the theoretical model in this experiment. First, each school assigns a priority to students that have a sibling in the school (over other students), and ties are resolved between students with equal priorities only when necessary (in our model all students had equal priorities). Second, students' preferences are not generated randomly, but taken as given from the data.

We run Monte Carlo simulations of students DA with STB and MTB using the rankings the students reported and the real priorities schools assigned to students. Many of the schools in the district contained multiple different programs that students rank separately, for consistency we will refer to each program as a distinct school.

For two tie-breaking rules A and B we say that A dominates B at a given school if for every $k \geq 1$, the computed expected number of students assigned to the school with rank at most k under A is at least the expected number under B minus one. We say B dominates A if the reverse is true, and we say A is equivalent to B if each dominates the other.

For the purpose of these experiments we slightly abuse the definition of popularity and refer to a school as *popular* if more students rank it their top choice than the school has capacity. Approximately 34% of the schools were popular.

We simulated the DA algorithm with both STB and MTB generated each 1000 times. By taking the average over the resulting rank distributions, we estimated the distribution of ranks of students assigned to each school. Table 1 gives the fraction of popular and non-popular schools for which their rank distribution under STB strictly rank-dominates MTB, both STB and MTB dominate each other (denoted as STB = MTB), MTB strictly dominates STB, and neither dominates the other.

Table 1: Frequency of dominance between STB and MTB in schools

	Popular schools	Non-popular schools
STB strictly dominates MTB	80%	16%
STB $\equiv$ MTB	18%	40%
MTB strictly dominates STB	2%	36%
Neither dominates	0%	8%

6 Extensions

The result of the first part of Theorem 1 can be extended to apply to a much broader set of distributions of student preferences, when the mass of students relative to the capacity of the schools is sufficiently large. The distribution of student preferences needs only satisfy a minor technical condition we call *non-ordered*.

Definition 8. *A school choice problem has non-ordered student preferences if there is no school $j < n$ such that the full mass of students prefer j to all schools $j' > j$.*

Theorem 2. *For any school choice problem $C = (m, q, N)$ with non-ordered student preferences, there exists $N' \in \mathbb{R}$ such that if $N \geq N'$, STB dominates MTB for students with any preference order.*

The following corollary relaxes the non-ordered condition of Theorem 2 at the expense of a slightly weaker result. First, a few definitions are needed. For fixed measure m and capacities q, define $R_P^{MTB}(N)$ to be R_P^{MTB} for school choice problem $C = (m, q, N)$. Define

$$D_P = \lim_{N' \to \infty} R_P^{MTB}(N'),$$

which is the distribution of ranks of students with preferences P under MTB in the limit as the mass of students approaches infinity.

Corollary 2.1. *For any school choice problem with n schools, for all $P \in \Pi_n$*

$$R_P^{STB} \succeq D_P.$$

The proofs of Theorem 2 and Corollary 2.1 are given in Appendices A.4 and A.5.

7 Summary

This paper considered the problem of resolving ties when assigning students to schools with heterogeneous qualities using the deferred acceptance mechanism. It is shown that when schools are "popular," a single lottery used by all schools is preferable to having each school using a separate lottery, in a first stochastic order sense, for all students' ex-ante and all schools. We conjecture that such dominance holds in all popular schools even when the set of possible schools include non-popular schools.

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A Proofs

A.1 Proof of Proposition 1

Consider a school choice problem $C = (m, q, N)$ with $n = 3$ and satisfying MNL preferences. If every school is popular, then by Theorem 1, STB dominates MTB at every school. We now show that regardless of whether school 1 or 2 is popular, STB dominates MTB at these schools. First, since STB only assigns rank 1 students to school 1, STB dominates MTB at school 1. Secondly we show that STB dominates MTB at school 2. In the first round of DA, the same mass of students applies to school 2 under STB as in MTB. In the second round, under STB only students who were rejected from school 1 in round 1 will apply to school 2. On the other hand, under MTB students who were rejected from school 1 in round 1 will apply to school 2 in round 2, in addition to possible students who were rejected from school 3. So, at the end of round 2, more students have applied to school 2 who rank it at least rank 2 under MTB than under STB. Under STB, no students will apply to school 2 in any future round. Under MTB, students will continue applying to school 2 for additional rounds. At the end of each round, the mass of students who have applied to school 2 who rank it at least rank 2 weakly increases. So, fewer students who rank school 2 at least rank 2 apply to school 2 under STB than under MTB. Furthermore, students under STB have a disadvantage when applying to schools in the second round because they do not receive a new lottery number. Thus, STB assigns fewer students with rank at least 2 to school 2 than MTB does. Since STB assigns no students with rank greater than 2 to school 2, STB dominates MTB at school 2. □

A.2 Proof of Lemma 3

Let A_j be the set of students who prefer j to all schools $k > j$, and see that all students assigned to j under both VMTB(β^0) and STB are in A_j. Recall that $m(A_j)$ denotes the probability that a randomly chosen student is in A_j. For $k \in [n]$, let Q_j^k denote the set of students for whom j is their kth choice. We construct a stochastic vector W of length j as follows. Let $W(1) = R_j^{STB}(1)$. For each $k \in [2, ..., j]$ let

$$W(k) = d \cdot m(A_j \cap Q_j^k),$$

for some constant d, and for $k > j$ let $W(k) = 0$.

First, we show that $R_j^{STB} \succeq W$. For a stochastic vector D of length n and constant

$k \in [n]$, we define the simplifying notation

$$P(D \geq k) = \sum_{p=k}^{n} D(p).$$

We also define

$$\tilde{Q}_j^k = \bigcup_{p=k}^{n} Q_j^k.$$

Let M_j be the set of students assigned to j under STB. Since $M_j \subseteq A_j$, for any $k \in [j]$ we have

$$P(R_j^{STB} \geq k) = \frac{m(M_j \cap \tilde{Q}_j^k)}{m(M_j)} = \frac{m(M_j|A_j \cap \tilde{Q}_j^k)m(A_j \cap \tilde{Q}_j^k)}{m(M_j)}, \tag{8}$$

and

$$P(W \geq k) = d\sum_{p=k}^{j} m(A_j \cap Q_j^p) = d \cdot m(A_j \cap \tilde{Q}_j^k). \tag{9}$$

Next, since $W(1) = R_j^{STB}(1)$ we have $P(W \geq 2) = P(R_j^{STB} \geq 2)$, so

$$d \cdot m(A_j \cap \tilde{Q}_j^2) = \frac{m(A_j \cap \tilde{Q}_j^2)m(M_j|A_j \cap \tilde{Q}_j^2)}{m(M_j)}$$

and thus

$$d = \frac{m(M_j|A_j \cap \tilde{Q}_j^2)}{m(M_j)}. \tag{10}$$

Then from equations (8), (9) and (10) we get

$$\frac{P(W \geq k)}{P(R_j^{STB} \geq k)} = \frac{d \cdot m(M_j)}{m(M_j|A_j \cap \tilde{Q}_j^k)} = \frac{m(M_j|A_j \cap \tilde{Q}_j^2)}{m(M_j|A_j \cap \tilde{Q}_j^k)}.$$

The following claim then implies that $P(W \geq k) \geq P(R_j^{STB} \geq k)$ for all $k \in [n]$, and hence $R_j^{STB} \succeq W$.

Claim 3. *For any $k \in [2, ..., j]$,*

$$m(M_j|A_j \cap \tilde{Q}_j^k) \leq m(M_j|A_j \cap \tilde{Q}_j^2).$$

Proof. Fix a value of $k \in [2, ..., j]$. Fix a lottery number $L \geq \alpha_j$, and let i be a randomly chosen student with lottery number L. Let $P = (P_1, ..., P_n)$ denote the preferences of i. Let

i' be a randomly chosen student in A_j with lottery number L, who does not rank j her top choice. Let B_L be the set of schools in $[j-1]$ with STB cutoffs above L. Then for each school $s \in [j-1]$,

$$\begin{aligned} P(P'_1 = s) = P(P_1 = s | i \in A_j \cap \tilde{Q}_j^2) &= \frac{P(i \in A_j \cap \tilde{Q}_j^2 | P_1 = s) P(P_1 = s)}{P(i \in A_j \cap tildeQ_j^2)} \\ &= \frac{P(i \in A_j | P_1 = s) P(P_1 = s)}{P(i \in A_j \cap \tilde{Q}_j^2)} \\ &= \frac{P(i \in A_j | P_1 = s) P(P_1 = s)}{\sum_{p=1}^{j-1} P(i \in A_j | P_1 = p) P(P_1 = p)}. \end{aligned} \tag{11}$$

The value of $P(i \in A_j)$ and $P(i \in A_j | P'_1 = p)$ can be determined as follows. When a student's preferences are being drawn from the multinomial-logit model, her first choice is drawn first, then her second choice, and so on. When a school in $\{j, ..., n\}$ is first drawn, the probability that school j is drawn is

$$\frac{\mu_j}{\sum_{p=j}^{n} \mu_p}.$$

Thus,

$$P(i \in A_j) = \frac{\mu_j}{\sum_{p=j}^{n} \mu_p}, \tag{12}$$

and by the same argument, for any school $k' \in [j-1]$,

$$P(i \in A_j | P_1 = k') = \frac{\mu_j}{\sum_{p=j}^{n} \mu_p}.$$

So from (11) we get

$$\begin{aligned} P(P'_1 = s) &= \frac{P(P_1 = s)}{\sum_{p=1}^{j-1} P(P_1 = p)} \\ &= \frac{\mu_s}{\sum_{p=1}^{j-1} \mu_p}. \end{aligned} \tag{13}$$

Observe that for any $p \in B_L$,

$$P(i' \in M_j | P'_1 = p) = P(i \in M_j | i \in A_j). \tag{14}$$

This follows from the independence of irrelevant alternatives of the MNL preference model.

So, from (13) and (14) we get

$$
\begin{aligned}
P(i' \in M_j) &= \sum_{p \in B_L} P(i' \in M_j | P'_1 = p) P(P'_1 = p) \\
&= \sum_{p \in B_L} P(i \in M_j | i \in A_j) P(P'_1 = p) \\
&= P(i \in M_j | i \in A_j) \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}.
\end{aligned} \tag{15}
$$

Now, let i^* be a randomly chosen student in $A_j \cap \tilde{Q}_j^k$ with lottery number L, and preferences $P^* = (P_1^*, ..., P_n^*)$. We define the notation

$$P_{[p]} = \{P_1, ..., P_p\}$$

for $p \in [n]$. See that if $i^* \in M_j$, then $P_{[k-1]} \subseteq B_L$. So,

$$
\begin{aligned}
P(i^* \in M_j) = P(i \in M_j | i \in A_j \cap \tilde{Q}_j^k) = P(P_{[k-2]} \subseteq B_L | i \in A_j \cap \tilde{Q}_j^k) \\
\times\ P(P_{[k-1]} \in B_L | i \in A_j \cap \tilde{Q}_j^k, P_{[k-2]} \subseteq B_L) \\
\times\ P(i \in M_j | i \in A_j).
\end{aligned} \tag{16}
$$

For every set $G \subseteq B_L$ such that $|G| = k-2$, let

$$p(G) = P(P^*_{[k-2]} = G | P^*_{[k-2]} \subseteq B_L).$$

Then

$$
\begin{aligned}
P(P_{[k-1]} \in B_L | i \in A_j \cap \tilde{Q}_j^k, P_{[k-2]} \subseteq B_L) &= \sum_{G \subseteq B_L, |G| = k-2} p(G) \frac{\sum_{p \in B_L} \mu_p - \sum_{p \in G} \mu_p}{\sum_{p=1}^{j-1} \mu_p - \sum_{p \in G} \mu_p} \\
&\leq \sum_{G \subseteq B_L} p(G) \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p} \\
&= \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}
\end{aligned} \tag{17}
$$

and thus from (16),

$$\begin{aligned} P(i^* \in M_j) &\leq P(P_{[k-2]} \subseteq B_L | i \in A_j \cap \tilde{Q}_j^k) \frac{\sum_{p \in B_L} \mu_p}{P(i \in M_j | i \in A_j)} \sum_{p=1}^{j-1} \mu_p. \\ &\leq P(i \in M_j | i \in A_j) \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}. \end{aligned} \tag{18}$$

By (15) and (18) we obtain

$$P(i^* \in M_j) \leq P(i' \in M_j). \tag{19}$$

Finally, if $L < \alpha_j$, then

$$P(i' \in M_j) = P(i^* \in M_j) = 0.$$

So, $P(i' \in M_j) \leq P(i^* \in M_j)$ for all $L \in [0, 1]$, and we have proven Claim 3. □

Next, it needs to be shown that $W \succeq V_j$. Fix a student i with preferences $P = (P_1, ..., P_n)$. If $i \in A_j$, $P_j = j$ and $j \in M_j$, then $P_{[j-1]} = [j-1]$ so i must have been rejected by every school in $[j-1]$. Let

$$K = \frac{V_j(j)}{P(A_j \cap Q_j^j) \prod_{p=1}^{j-1} \beta_p},$$

so

$$V_j(j) = K \cdot P(A_j \cap Q_j^j) \prod_{p=1}^{j-1} \beta_p.$$

If $i \in A_j$ and $P_k = j$ for some $k \leq j$, then i only needs to be rejected by a subset of the schools in $[j-1]$ for her to apply to school j. Therefore for $k \in [j-1]$,

$$V_j(k) \geq K \cdot P(A_j \cap Q_j^k) \prod_{p=1}^{j-1} \beta_p.$$

Recall that for all $p \in \{2, ..., j\}$,

$$\frac{W(p)}{P(A_j \cap Q_j^p)} = d.$$

Thus, if it is shown that

$$d \leq K \cdot \prod_{p=1}^{j-1} \beta_p, \tag{20}$$

then $W(k) \leq V_j(k)$ for all $2 \leq k \leq j$.. It will then follow that $W \succeq V_j$ by Claim 2. So, it remains to show that inequality (20) holds. For all $k \in [j-1]$ we have

$$V_j(k) \leq K \cdot m(A_j, Q_j^k)$$

and

$$\sum_{p=1}^{j} V_j(p) = 1.$$

So,

$$1 = \sum_{p=1}^{j} V_j(p) \leq K \sum_{p=1}^{j} m(A_j \cap Q_j^p) = K \cdot m(A_j)$$

and thus

$$K \geq \frac{1}{m(A_j)}.$$

Proposition 3 gives that

$$\prod_{p=1}^{j-1} \beta_p \geq 1 - \frac{\sum_{i=1}^{j-1} q_p}{N \sum_{p=1}^{j-1} \mu_p},$$

and so

$$K \cdot \prod_{p=1}^{j-1} \beta_p \geq \frac{1}{m(A_j)}(1 - \frac{\sum_{i=1}^{j-1} q_p}{N \sum_{p=1}^{j-1} \mu_p}). \tag{21}$$

Next we need an upper bound for d. Observe that

$$\begin{aligned} 1 &= \sum_{p=1}^{j} W(p) \\ &= W(1) + \sum_{p=2}^{j} W(p) \qquad (22) \\ &= R_j^{STB}(1) + d \sum_{p=2}^{j} m(A_j \cap Q_j^p). \end{aligned}$$

The mass of students who rank school j as their first choice is $N\mu_j$, and these students are accepted to school j with probability $1 - \alpha_j$. Since the total mass of students assigned to school j is q_j, we have

$$R_j^{STB}(1) = \frac{N\mu_j(1-\alpha_j)}{q_j}. \tag{23}$$

Moreover,

$$\begin{aligned}\sum_{p=2}^{j} m(A_j \cap Q_j^p) &= m(A_j, \tilde{Q}_j^2) \\ &= m(A_j) - m(A_j \cap Q_j^1) \\ &= m(A_j) - \mu_j.\end{aligned} \tag{24}$$

Then (22), (23) and (24) give

$$1 = N\frac{\mu_j}{q_j}(1-\alpha_j) + (m(A_j) - \mu_j)d. \tag{25}$$

Let $r_j = \sum_{p=j}^{n} \mu_p$. Then by (12),

$$m(A_j) = \frac{\mu_j}{r_j}.$$

From Proposition 1,

$$1 - \alpha_j = \frac{1}{N}(\sum_{p=1}^{j-1} q_p + \frac{r_j q_j}{\mu_j}).$$

Thus, equation (25) becomes

$$\begin{aligned}1 &= \frac{\mu_j}{q_j}\sum_{p=1}^{j-1} q_p + r_j + (\frac{\mu_j}{r_j} - \mu_j)d \\ &= \frac{\mu_j}{q_j}\sum_{p=1}^{j-1} q_p + r_j + \frac{\mu_j}{r_j}(1 - r_j)d\end{aligned}$$

which implies

$$\begin{aligned}d &= \frac{r_j}{\mu_j} \cdot \frac{1}{1 - r_j}(1 - r_j - \frac{\mu_j}{q_j}\sum_{p=1}^{j-1} q_p) \\ &= \frac{r_j}{\mu_j}(1 - \frac{\mu_j}{q_j(1 - r_j)}\sum_{p=1}^{j-1} q_p).\end{aligned}$$

Since school j is popular, $N\mu_j \geq q_j$, so

$$\begin{aligned} d &\leq \frac{r_j}{\mu_j}(1 - \frac{\sum_{p=1}^{j-1} q_p}{N(1-r_j)}) \\ &= \frac{1}{m(A_j)}(1 - \frac{\sum_{p=1}^{j-1} q_p}{N(1-r_j)}). \end{aligned} \tag{26}$$

By (21) and (26) we obtain (20), which gives $V_j \succeq W$. This concludes the proof of Lemma 3. $\square$

A.3 Proof of Lemma 4

The proof makes use of the following definitions. Recall that we fixed a randomly chosen student i with preferences $(P_1, ..., P_n)$.

Definition 9. *For a set of cutoffs β^* and $k \in [n]$, let $H_k^{\beta^*}$ be the event that student i is assigned to school k under VMTB(β^*).*

Definition 10. *For $k \in [n]$, let q_k^i be the rank of school k in i's preference order.*

Now, suppose that $\beta_{j'} = 0$ and $\beta'_{j'} = 1$. The following claim shows a convenient reformulation of the problem.

Claim 4. *Suppose that for all $k \in [n]$,*

$$P(q_{j'}^i < q_j^i | H_j^{\beta'}, q_j^i \geq k) \geq P(q_{j'}^i < q_j^i | H_j^{\beta'}). \tag{27}$$

Then $R_j^{\beta} \succeq R_j^{\beta'}$.

Proof. Suppose inequality (27) holds for all $k \in [n]$. It needs to be shown that for all $k \in [n]$,

$$\sum_{p=k}^{n} R_j^{\beta'}(p) \geq \sum_{p=k}^{n} R_j^{\beta}(p).$$

Consider initially assigning students according to VMTB(β'). This initial assignment can be transformed to an assignment according to VMTB(β), by lowering the cutoff for school j' to 0, and any student who prefers j' to her initial assignment becomes reassigned to j'. Then if i was initially assigned to school j with rank p, she will be reassigned to school j' with probability

$$P(q_{j'}^i < q_j^i | H_j^{\beta'}, i_j = p).$$

Thus, for all $p \in [n]$,

$$R_j^\beta(p) = C \cdot (1 - P(q_{j'}^i < q_j^i | H_j^{\beta'}, i_p = j)) \cdot R_j^{\beta'}(p),$$

where C is a normalizing constant so that R_j^β has total mass of one. For $k \in [n]$, conditioned on i being assigned to school j and having rank at least k, we have

$$\sum_{p=k}^{n} R_j^\beta(p) = C \cdot (1 - P(q_{j'}^i < q_j^i | H_j^{\beta'}, q_j^i \geq k)) \cdot \sum_{p=k}^{n} R_j^{\beta'}(p).$$

Setting $k = 1$ in the above equation, we get

$$\sum_{p=1}^{n} R_j^\beta(p) = C \cdot (1 - P(q_{j'}^i < q_j^i | H_j^{\beta'})) \cdot \sum_{p=1}^{n} R_j^{\beta'}(p).$$

Since

$$\sum_{p=1}^{n} R_j^\beta(p) = \sum_{p=1}^{n} R_j^{\beta'}(p) = 1,$$

this gives

$$C = \frac{1}{1 - P(q_{j'}^i < q_j^i | H_j^{\beta'})}.$$

So, for $k \in [n]$, by inequality (27)

$$\begin{aligned}\sum_{pkm}^{n} R_j^\beta(p) &= \frac{1 - P(q_{j'}^i < q_j^i | H_j^{\beta'}, q_j^i \geq k)}{1 - P(q_{j'}^i < q_j^i | H_j^{\beta'})} \cdot \sum_{p=k}^{n} R_j^{\beta'}(p) \\ &\geq \sum_{p=k}^{n} R_j^{\beta'}(p).\end{aligned}$$

Thus $R_j^\beta \succeq R_j^{\beta'}$. □

The next step is to prove that inequality (27) holds for all $k \in [n]$. Consider the exponential clock method of drawing student preferences described in Claim 1, where for each $p \in [n]$, X_p is the time that clock p rings and $X^{(p)}$ is the pth earliest clock to ring. Then inequality (27) is equivalent to

$$P(X_{j'} < X_j | H_j^{\beta'}, X_j \geq X^{(k)}) \geq P(X_{j'} < X_j | H_j^{\beta'}). \tag{28}$$

We will now show that inequality (28) holds for all $k \in [n]$. For $k \in [n-1]$ and $S' \subseteq S$ where $|S'| = k$ and $j, j' \notin S'$, let

$$p(S') = P(i_{[k]} = S' | H_j^{\beta'}, q_j^i \geq k+1).$$

That is, $p(S')$ is the probability that i's top k ranked schools are S', conditional on i being assigned to j with rank at least $k+1$. For a set of schools $B \subseteq S$ where $j', j \notin B$, we define a set of cutoffs β^B as follows: for all schools $p \in B$, $(\beta^B)_p = 1$, and for all schools $p \notin B$, $(\beta^B)_p = \beta'_p$. Note that VMTB(β^B) is equivalent to VMTB(β') where schools in B have been removed from the school choice problem. Let

$$P_B = P(X_{j'} < X_j | H_j^{\beta^B}).$$

Observe that if i is assigned to j with rank at least k, then she must have been rejected by each school in $i_{[k-1]}$. If $j' \in i_{[k-1]}$, then the probability that i prefers j' to j is one. Otherwise, the probability i prefers j' to j will be $P_{i_{[k-1]}}$. Let

$$b_k = P(j' \in i_{[k-1]} | H_j^{\beta'}, X_j \geq X^{(k)}).$$

Then

$$P(X_i < X_j | H_j^{\beta'}, X_j \geq X^{(k)}) = b_k + (1 - b_k) \cdot \sum_{S' \subseteq S : |S'| = k-1, j, j' \notin S'} p(S') P_{S'}. \tag{29}$$

The following proposition, along with equation (29) gives inequality (28).

Proposition 4. *For all $S' \subseteq S$ where $j, j' \notin S'$,*

$$P_{S'} \geq P(X_{j'} < X_j | H_j^{\beta'}).$$

Proof. Let $\tilde{X}_j$ be a random variable with distribution equal to the distribution of X_j conditional on $H_j^{\beta'}$. Since $X_{j'}$ is an exponential random variable with rate $\mu_{j'}$, the cdf of $X_{j'}$ is

$$F(x) = 1 - e^{-\mu_{j'} x}.$$

Since $\beta'_{j'} = 1$, the value of $X_{j'}$ does not affect the assignment of i, so $X_{j'}$ is independent of

$H_j^{\beta'}$. Furthermore, since $X_{j'}$ is independent of X_j, we have

$$P(X_i < X_j | H_j^{\beta'}) = P(X_i < \tilde{X}_j) = E_{\tilde{X}_j}[F(x)].$$

For a set of schools $B \subseteq S$ where $j, j' \notin B$, let $\tilde{X}_j^B$ be a random variable with distribution equal to the distribution of X_j conditional on $H_j^{\beta^B}$. Then

$$P_{S'} = P(X_i < \tilde{X}_j^{S'}) = E_{\tilde{X}_j^{S'}}[F(x)].$$

Since $F(x)$ is an increasing function, if $\tilde{X}_j \succeq \tilde{X}_j^{S'}$ then this will imply

$$E_{\tilde{X}_j^{S'}}[F(x)] \geq E_{\tilde{X}_j}[F(x)],$$

which gives the proposition. So, it remains to show $\tilde{X}_j \succeq \tilde{X}_j^{S'}$, which will follow from the next claim. The claim implies that for any $B_1 \subseteq B_2 \subseteq S$ where $j, j' \in B_1 \cap B_2$, we have $\tilde{X}_j^{B_1} \succeq \tilde{X}_j^{B_2}$. First, fix an indexing of the schools excluding j and j',

$$S \setminus \{i, j\} = \{a_1, a_2, ..., a_{n-2}\}.$$

For $p \in \{0, 1, ..., n-2\}$, let

$$\tilde{X}_j^p = \tilde{X}_j^{\{a_1, \ldots, a_p\}}.$$

Claim 5.

$$\tilde{X}_j^0 \succeq \tilde{X}_j^1 \succeq ... \succeq \tilde{X}_j^{n-2}.$$

Proof. For $p \in \{0, 1, ..., n-2\}$, let $f_p(x)$ be the pdf of $\tilde{X}_j^p$. Consider assigning i by VMTB($\beta^{\{a_1, \ldots, a_{n-2}\}}$). Since $\beta_p^{\{a_1, \ldots, a_{n-2}\}} = 1$ for all schools $p \neq j$, conditional on any value of X_j we have that i will be assigned to school j with probability $1 - \beta'_j$. Thus, the distribution of $\tilde{X}_j^{n-2}$ is equal to the distribution of X_j, so

$$f_{n-2}(x) = \mu_j e^{-\mu_j x}.$$

Hence, $f(x)$ is decreasing over $x \geq 0$. Now, as an inductive hypothesis, assume that for some $p \in \{1, 2, ..., n-2\}$, $f_p(x)$ is decreasing over $x \geq 0$. We will show that this implies that both $f_{p-1}(x)$ is decreasing over $x \geq 0$ and that $\tilde{X}_j^{p-1} \succeq \tilde{X}_j^p$. Now, suppose that i is assigned to j by VMTB($\beta^{\{a_1, \ldots, a_p\}}$, and $X_j = x$. Since $\beta_{a_p}^{\{a_1, \ldots, a_p\}} = 1$, the probability that i prefers school

a_p to j is

$$F_p(x) = P(X_{a_p} < x) = 1 - e^{-\mu_{a_p} x}.$$

Consider now lowering the cutoff for school a_p to β'_{a_p}, and reassigning students with high enough lottery numbers to school a_p if they prefer it to their initial assignment. Observe that this process will result in the assignment under VMTB($\beta^{\{a_1,\ldots,a_{p-1}\}}$). Then for some positive normalizing constant $K' > 0$,

$$f_{p-1}(x) = K' f_p(x)[F_{p-1}(x)(1 - \beta'_{a_{p-1}}) + (1 - F_{p-1}(x)].$$

Since $1 - \beta'_{a_{p-1}} \leq 1$ and $F_{p-1}(x)$ is increasing in x, we have that

$$K'[F_{p-1}(x)(1 - \beta'_{a_{p-1}}) + (1 - F_{p-1}(x))]$$

is decreasing in x for $x \geq 0$. Since by hypothesis $f_p(x)$ is decreasing for $x \geq 0$, we have $f_{p-1}(x)$ is decreasing for $x \geq 0$. To show $\tilde{X}_j^{p-1} \succeq \tilde{X}_j^p$ we need the following claim.

Claim 6. *Let Y be a nonnegative random variable with decreasing pdf $f(x)$, and Z be a nonnegative random variable with pdf $h(x) = f(x)g(x)$, where $g(x)$ is a nonnegative decreasing function. Then $Z \succeq Y$.*

Proof. By definition, $Z \succeq Y$ is equivalent to

$$\int_0^t f(x)dx \leq \int_0^t h(x)dx, \ \forall t \geq 0. \tag{30}$$

Since

$$\int_0^\infty f(x)dx = \int_0^\infty h(x)dx = 1,$$

we get that (30) is equivalent to

$$\int_t^\infty f(x)dx \geq \int_t^\infty h(x)dx, \ \forall t \geq 0. \tag{31}$$

By (30) and (31) we obtain that $Y \succeq Z$ if

$$\frac{\int_0^t f(x)dx}{\int_t^\infty f(x)dx} \leq \frac{\int_0^t h(x)dx}{\int_t^\infty h(x)dx}.$$

Since $g(x)$ is decreasing,

$$\frac{\int_0^t h(x)dx}{\int_t^\infty h(x)dx} = \frac{\int_0^t f(x)g(x)dx}{\int_t^\infty f(x)g(x)dx} \geq \frac{\int_0^t f(x)g(t)dx}{\int_t^\infty f(x)g(t)dx} = \frac{\int_0^t f(x)dx}{\int_t^\infty f(x)dx},$$

which gives Claim 6. $\square$

Using Claim 6 with $Y = \tilde{X}_j^p, Z = \tilde{X}_j^{p-1}, f(x) = f_p(x)$ and

$$g(x) = K[F_{p-1}(x)(1 - \beta_{a_{k+1}}) + (1 - F_{p-1}(x)],$$

we get that if $\tilde{X}_j^p$ has a decreasing pdf, then $\tilde{X}_j^{p-1} \succeq \tilde{X}_j^p$. This completes the proof of Claim 5, and the proof of Proposition 4. $\square$

$\square$

Lemma 4 has now been proven in the special case that $\beta_{j'} = 0$ and $\beta'_{j'} = 1$. It remains to show that the lemma holds in the general case. Now, suppose β and β' satisfy $\beta_{j'} \leq \beta'_{j'}$ and $\beta_p = \beta'_p$ for all $p \neq j'$. Let e_i be the vector with one in the ith entry and zero in the other entries. See that $\beta - \beta_i e_i$ is equal to β, but with the ith entry set to zero. Similarly, $\beta + (1 - \beta_i)e_i$ is equal to β but with the ith cutoff set to one. Let

$$E = R_j^{\beta - \beta_i e_i}$$

and

$$F = R_j^{\beta + (1-\beta_i)e_i}.$$

By the special case of the lemma, we have that $E \succeq F$. Now, consider assigning students by VMTB(β) according to the following equivalent process. Each student is independently put into case 1 with probability $\beta_{j'}$, and put into case 2 with probability $1 - \beta_{j'}$. Then, students in case 1 are assigned according to VMTB($\beta - \beta_i e_i$), and students in case 2 are assigned according to VMTB($\beta + (1 - \beta_i)e_i$). The case 1 students correspond to the students who have a lottery number for school j' below $\beta_{j'}$, and the case 2 students to the students who have a lottery number above $\beta_{j'}$. Thus, this process is indeed equivalent to VMTB(β). Let c_1 be the probability that a randomly chosen case 1 student is assigned to school j, and let c_2 be the probability that a randomly chosen case 2 student is assigned to school j. Then, for some normalizing constant $C' > 0$,

$$R_j^{\beta} = C'[(1-\beta_{j'})c_1 E + \beta_{j'} c_2 F]. \tag{32}$$

See that for some $\lambda_1 \in [0,1]$,

$$R_j^{\beta} = \lambda_1 E + (1-\lambda_1)F. \tag{33}$$

To solve for λ_1, from (32) we obtain

$$\lambda_1 = C'(1-\beta_{j'})c_1$$

and

$$1-\lambda_1 = C'\beta_{j'}c_2.$$

Adding the above two equations together and solving for C' gives

$$C' = \frac{1}{(1-\beta_{j'})c_1 + \beta_{j'}c_2}.$$

Thus

$$\lambda_1 = \frac{(1-\beta_{j'})c_1}{(1-\beta_{j'})c_1 + \beta_{j'}c_2}.$$

Similarly, for some normalizing constant C'',

$$R_j^{\beta'} = C''[(1-\beta'_{j'})c_1 E + \beta'_{j'} c_2 F],$$

and so

$$R_j^{\beta'} = \lambda_2 E + (1-\lambda_2)F \tag{34}$$

where

$$\lambda_2 = \frac{(1-\beta'_{j'})c_1}{(1-\beta'_{j'})c_1 + \beta'_{j'}c_2}.$$

Since $\beta'_{j'} \geq \beta_{j'}$ we then have that $\lambda_1 \geq \lambda_2$. Finally, from (33) and (34) we get that for any

$k \in [n]$,

$$\sum_{p=1}^{k} R_j^{\beta}(p) - \sum_{p=1}^{k} R_j^{\beta'}(p) = (\lambda_1 - \lambda_2) \sum_{p=1}^{m} E(p) + (\lambda_2 - \lambda_1) \sum_{p=1}^{m} F(p)$$
$$= (\lambda_1 - \lambda_2)[\sum_{p=1}^{m} E(p) - \sum_{p=1}^{m} F(p)]$$
$$\leq 0,$$

where the inequality follows from $\lambda_1 \geq \lambda_2$ and $E \succeq F$. Thus, $R_j^{\beta} \succeq R_j^{\beta'}$ as desired, which concludes the proof of Lemma 4.

A.4 Proof of Theorem 2

Consider a school choice problem $C = (m, q, N)$. Fix a preference order $P = (P_1, ..., P_n) \in \Pi_n$ such that $m(P) > 0$, and fix a rank $k \in [n]$. If $j = P_p$, we write $P^{-1}(j) = p$. Index the schools such that for all schools j and j', if $\alpha_j = \alpha_{j'}$ and $P^{-1}(j) < P^{-1}(j')$, then $j > j'$. Let

$$R_{P,k}^{STB} = \sum_{p=1}^{k} R_P^{STB}(p)$$

and

$$R_{P,k}^{MTB} = \sum_{p=1}^{k} R_P^{MTB}(p).$$

We need to show that

$$R_{P,k}^{MTB} \leq R_{P,k}^{STB}$$

for any sufficiently large N. First, we show an upper bound for $R_{P,k}^{MTB}(P, N)$. Let $Q = \sum_{j=1}^{n} q_j$. Observe that

$$R_{P,n}^{MTB} = R_{P,n}^{STB} = \frac{Q}{N},$$

so we assume that $k < n$. Let $\beta^N = (\beta_1^N, ..., \beta_n^N)$ be the MTB cutoffs for C. Since $\prod_{j=1}^{n} \beta_j^N = 1 - \frac{Q}{N}$, it must be that for all $j \in [n]$,

$$\lim_{N \to \infty} \beta_j^N = 1.$$

Now, for a fixed student i, we say that school j is *available* to i if i has priority for j at least as large as the cutoff for j. Then i will be assigned to a school iff at least one school is available for her. For each school j, let B_j^N be the event that j is the only school available to i, as a function of N. Let A^N be the event that no school is available to i, as a function of N. Then

$$P(A^N) = 1 - \frac{Q}{N}$$

and for all $j \in [n]$,

$$P(B_j^N) \geq (1 - \beta_j^N) P(A^N). \tag{35}$$

During the DA algorithm at most a mass of N students will apply to any given school. For N sufficiently large, every school will be filled to capacity. This implies that all $j \in [n]$

$$1 - \beta_j^N \geq \frac{q_j}{N}. \tag{36}$$

Then by (35) and (36) we get

$$P(B_j^N) \geq \frac{q_j}{N} P(A^N). \tag{37}$$

Note that if the event B_j^N occurs, then i will be assigned to j regardless of her preferences. Let

$$E = \bar{A}^N \backslash (\bigcup_{j=1}^{k} B_j^N),$$

that is, E is the event that at least two schools are available to i. Because the events $B_1^N, ..., B_n^N$ are all disjoint and contained in A^N,

$$\begin{aligned} P(E) &= P(\bar{A}^N) - \sum_{j=1}^{n} P(B_j^N) \\ &\leq \frac{Q}{N} - \sum_{j=1}^{n} \frac{q_i}{N} P(A^N) \\ &= \frac{Q}{N}(1 - P(A^N)) \\ &= \frac{Q^2}{N^2}, \end{aligned}$$

where the second line follows from (37). Since a mass of $N - Q$ students is unassigned, at least $N - Q$ students apply to every school. So for all $j \in [n]$

$$(N - Q)(1 - \beta_j^N) \leq q_j$$

and thus

$$1 - \beta_j^N \leq \frac{q_i}{N - Q}. \tag{38}$$

For all $j \in [n]$,

$$P(B_j^N) \leq 1 - \beta_j^N. \tag{39}$$

Then (38) and (39) give that

$$P(B_j^N) \leq \frac{q_j}{N - Q}.$$

Letting $Q_k = \sum_{j=1}^{k} q_{P_j}$, we have

$$R_{P,k}^{MTB} \leq \sum_{j=1}^{k} P(B_{P_j}^N) + P(E)$$

$$\leq \frac{Q_k}{N - Q} + \frac{Q^2}{N^2}.$$

Let $r_k^N = \frac{Q_k}{N}$. Then we obtain the upper bound

$$R_{P,k}^{MTB} \leq r_k^N \cdot \frac{N}{N - Q} + \frac{Q^2}{N^2}. \tag{40}$$

Therefore

$$R_{P,k}^{MTB} \leq r_k^N + O(\frac{1}{N^2}).$$

Next we need to show a lower bound for $R_{P,k}^{STB}$. If a student is picked at random, she will be assigned to a school in $\{P_1, ..., P_k\}$ with probability r_k^N. For any $P' \in \Pi_n$, let $g_k^N(P, P')$ be the probability that a student with preferences P' is assigned to a school in $\{P_1, ..., P_k\}$. Because in the DA algorithm it is a dominant strategy for the students to submit their true preferences,

$$g_k^N(P, P) \geq g_k^N(P, P') \ \ \forall P' \in \Pi_n. \tag{41}$$

See that

$$r_k^N = \sum_{P' \in \Pi_n} m(P', [0,1]^n) g_k^N(P, P') \leq g_k(P, P) = R_{P,k}^{STB}. \tag{42}$$

Let $m_k = \max\{P_1, ..., P_k\}$. We know

$$R_{P,n}^{STB} = 1 - \alpha_n = \frac{Q}{N},$$

which for sufficiently large N is smaller than the upper bound for $R_{P,k}^{MTB}$ given by (40), since $Q_k < Q$. Now assume $m_k < n$. Because students' preferences are non-ordered, there is some $P^* \in \Pi_n$ and school $p > m_k$ such that $m(P^*, [0,1]^n) > 0$ and $(P^*)^{-1}(p) < (P^*)^{-1}(m_k)$. Note that by the definition of m_k, $P^{-1}(m_k) < P^{-1}(p)$ and $s \notin \{P_1, ..., P_k\}$. Furthermore, by our indexing of the schools, for all schools j such that $\alpha_j = \alpha_{m_k}$, $P^{-1}(m_k) < P^{-1}(j)$. Thus, a student with preferences P has a nonzero probability of being assigned to m_k under STB. That is,

$$R_P^{STB}(P^{-1}(m_k)) > 0.$$

Assume that $N \geq Q$. Then under STB, only students with a lottery number of at least $\frac{Q}{N}$ are assigned to a school. For a given student i, conditional on i having a lottery number of at least $\frac{Q}{N}$, the probability that i is assigned to any given school is independent of N. Thus, for some positive constant c,

$$R_P^{STB}(P^{-1}(m_k)) = \frac{cQ}{N}.$$

Now, consider two students i and i', where i has preferences P and i' has preferences P^*, and both students receive the same lottery number. Suppose their lottery number is such that i will be assigned to m_k, which happens with probability $\frac{cQ}{N}$. Then i' will not be assigned to m_k or any other school in $\{P_1, ..., P_k\}$, since i' prefers p to m_k and $\alpha_p \leq \alpha_{m_k}$. If the students receive a lottery number such that i is not assigned to m_k, i is still at least as likely as i' to be assigned to a school in $\{P_1, ..., P_k\}$. Thus,

$$g_k^N(P, P^*) \leq g_k^N(P, P) - \frac{cQ}{N}. \tag{43}$$

Combining (41), (42) and (43) we get the desired lower bound

$$R_{P,k}^{STB} \geq r_k^N + \frac{cQ}{N}(P^{-1}(m_k)) \cdot m(P^*, [0,1]^n). \tag{44}$$

The two bounds (40) and (44) give that $R_{P,k}^{MTB} \leq R_{P,k}^{STB}$ for N sufficiently large, which concludes the proof. $\square$

A.5 Proof of Corollary 2.1

Consider a school choice problem $C = (m, q, N)$. Fix a preference order $P = (P_1, ..., P_n) \in \Pi_n$ such that $m(P) > 0$, and fix a rank $k \in [n]$. Let

$$R^{STB}_{P,k} = \sum_{j=1}^{k} R^{STB}_{P}(k)$$

and

$$D_{P,k} = \sum_{j=1}^{k} D_P(k).$$

Then to prove the theorem it needs to be shown that

$$D_{P,k} \le R^{STB}_{P,k} \tag{45}$$

for all N. Let $Q = \sum_{j=1}^{n} q_j$, and $Q_k = \sum_{j=1}^{k} q_{P_j}$. From the proof of Theorem 2, inequality (44) gives that

$$R^{STB}_{P,k} \ge \frac{Q_k}{N}.$$

On the other hand, (40) gives that

$$D_{P,k} \le \frac{1}{N} \lim_{N' \to \infty} N'(\frac{Q_k}{N'} \cdot \frac{N'}{N' - Q} + \frac{Q^2}{(N')^2}) = \frac{Q_k}{N}.$$

So $D_{P,k} \le R^{STB}_{P,k}$, which completes the proof. $\square$

B Tightness of Results

The examples presented here demonstrate the necessity of the conditions in our theorems. Example 2 shows that STB may not dominate MTB at schools that are non-popular, even if the schools are over-demanded. By over-demanded, we mean that more students apply to the school than the school has capacity. In this example, in fact MTB dominates STB at school 3, and school 3 is an over-demanded but non-popular school. Note that in general, it is possible for neither STB nor MTB to dominate the other at a non-popular school.

Example 2 (Non-Dominance at Unpopular Schools)**.** *Consider the following school choice problem with* $n = 3$ *and satisfying MNL preferences. Let* $N = 4$, $\mu = (3, 2, 1)$, *and* $q = (1/3, 1/3, 1/3)$.

Claim 7. *In Example 2, STB does not dominate MTB at school 3.*

Proof. Observe that schools 1 and 2 are popular, but school 3 is non-popular. Because $N > \sum_{j=1}^{n} q_j$, school 3 is over-demanded. We compute the rank distributions at the schools to be (rounded to the nearest tenth)

$$R_1^{STB} = (1, 0, 0), R_2^{STB} = (0.8, 0.2, 0), R_1^{STB} = (0.5, 0.2, 0.3),$$

$$R_1^{MTB} = (0.9, 0.1, 0), R_2^{MTB} = (0.7, 0.3, 0), R_1^{MTB} = (0.5, 0.3, 0.2).$$

So, STB dominates MTB at schools 1 and 2, but not at school 3. $\square$

The next example shows why the non-ordered condition is necessary for Theorem 2.

Example 3 (necessity of non-ordered condition)**.** *Consider the following school choice problem* $C = (m, q, N)$ *with* $n = 3$. *Let* $N \geq \sum_{j=1}^{n} q_j$, *and let* m *be given by*

$$m((1, 2, 3)) = p, \;\; m((2, 3, 1)) = 1 - p,$$

where p *and* q *are any values such that* $\alpha_1 > \alpha_2 > \alpha_3$.

Claim 8. *In Example 3, STB does not dominate MTB for students with preferences* $(1, 2, 3)$.

Proof. Let $(R_P^{STB})_{\leq k}$ be the probability that a student with preferences P is assigned to a school in her top k choices under STB. Let $(R_P^{MTB})_{\leq k}$ be the same under MTB, and let $P^* = (1, 2, 3)$. We will show that

$$(R_{P^*}^{STB})_{\leq 2} < (R_{P^*}^{MTB})_{\leq 2}, \tag{46}$$

which gives that $R_{P^*}^{STB} \not\succeq R_{P^*}^{MTB}$. Under both STB and MTB, a randomly chosen student will be assigned to school 1 or 2 with probability

$$r = \frac{q_1 + q_2}{N}.$$

Under STB, a student will be assigned to school 1 or 2 iff she has a lottery number of at least α_2, regardless of her preferences. So, $(R_{P^*}^{STB})_{\leq 2} = r$. Now, fix a randomly chosen student i.

Under MTB, there is a nonzero probability that i has high enough lottery numbers to be accepted to both school 1 and 3, but not 2. In which case, i will be assigned to school 1 or 2 iff she has preferences P^*. Thus, a student with preferences P^* is strictly more likely to be assigned to school 1 or 2 than a student with preferences $(2, 3, 1)$. Therefore students with preferences P^* are assigned to one of their top two schools with probability strictly greater than r, which shows (46) as desired. $\square$

The final example shows that dominance of STB over MTB at every school does not hold for arbitrary distributions of student preferences, even in the limit as the mass of students grows.

Example 4 (no dominance at schools in the limit)**.** *Consider the following school choice problem $C = (m, q, N)$ with $n = 4$. Let $N > 1$ and $q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Let m be given by*

$$m((1, 2, 3, 4)) = p, \;\; m((4, 3, 2, 1)) = 1 - p,$$

where $p < 1$ is a constant sufficiently close to one so that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$.

Claim 9. *In Example 4,*

$$R_3^{STB}(N) \not\succeq R_3^{MTB}(N).$$

Proof. Because $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$, a student can only be assigned to school 3 under STB if the student has preferences $(1, 2, 3, 4)$. So,

$$R_3^{STB}(N) = (0, 0, 1, 0)$$

Since all students rank school 3 as their second or third choice, only students of rank two or three are assigned to school 3 under MTB. Because $N > 1$, under MTB a nonzero mass of students will be rejected from all schools, so every MTB cutoff is strictly greater than zero. Thus, a nonzero mass of students with rank 2 will be assigned to school 3 under MTB, and so

$$R_3^{MTB} = (0, c, 1 - c, 0)$$

for some constant $c > 0$. Thus $R_3^{STB} \not\succeq R_3^{MTB}$. $\square$