Persuading a Pessimist: Simplicity and Robustness

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Abstract

While in practice most persuasion mechanisms have a simple structure, optimal signals in the Bayesian persuasion framework may not be so. Many signals in practice have an “interval structure”, i.e. they partition an ordered state space into intervals. The interval structure of optimal signals has been studied in the Bayesian persuasion literature; e.g., [Ivanov 2015, Kolotilin 2017, Dworczak and Martini 2018] provide sufficient, and in some cases, necessary conditions for the optimality of such signals in different settings. Another studied structural property is “monotonicity”, which means that the optimal signal does not recommend lower actions at higher higher states when both of the action and state spaces are ordered [Mensch 2018].

We show that the optimal signal features both of the properties—the interval structure and monotonicity—when Receiver is a pessimist. A pessimistic receiver, rather than maximizing expected payoff, takes the action that promises the highest guaranteed level of payoff. This notion of pessimism can also be interpreted through the Maxmin Expected Utility model where Receiver is considered an expected utility maximizer who does not know the prior. Hence, our findings explain how the simplicity of optimal signals can emerge from ambiguity of the prior.

We also build on our approach to find optimal signals in the presence of upper quota constraints on the signal size, and to analyze the size of optimal signals in stochastic problem instances, where we find that optimal signals are typically small.

Keywords: Bayesian persuasion, information design, interval structure, monotone partitions, monotonicity, ambiguity aversion

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1 Introduction

Bayesian persuasion has become a standard framework for modeling information transmission in economics: Sender and Receiver share the same prior belief about the state of the world. Before the state of the world is realized, Sender constructs a signal, i.e., a distribution over signal realizations at each possible state of the world. After the state of the world is realized, a signal realization is relayed to Receiver. Receiver then takes action to maximize her expected utility which depends on her action and the state of the world, with the expectation taken with respect to her posterior after receiving the signal realization.

Most persuasion mechanisms in practice have a simple structure. In particular, many of the signals in practice have an “interval structure”, i.e., they partition an ordered state space into intervals. The interval structure of optimal signals has been studied in the Bayesian persuasion literature. For instance, [Ivanov 2015, Kolotilin 2017, Dworczak and Martini 2018] provide sufficient, and in some cases, necessary conditions for the optimality of such signals in different settings.

Another structural property that is observed in practice and has been studied separately in the literature is “monotonicity”, which means that the optimal signal does not recommend lower actions at higher states when both of the action and state spaces are ordered [Mensch 2018]. We show that the optimal signal features both of these properties—the interval structure and monotonicity—when Receiver is a pessimist. This notion of pessimism, defined more precisely next, is remarkably simple compared to the existing sufficient conditions for either of these properties.¹

We say Receiver is a pessimist when, rather than maximizing expected utility, she evaluates each action by the least preferred outcome that it can promise and takes the most promising action. This notion of pessimism can also be interpreted through the Maxmin Expected Utility model where Receiver is considered an expected utility maximizer who does not know the prior.² Hence, our findings explain how ambiguity of the prior can lead to the simplicity (and, as we will discuss, the robustness) of optimal signals.

When the action and the state space are ordered so that the interval structure and

¹The conditions in prior work often take the form of technical conditions involving payoff functions, as reviewed in Section 5. There, we also discuss that the prior work do not imply our findings.

²Such maxmin utility specifications are used in many applications to model decision-making when probabilities are ambiguous and not objectively known. For example, they have been used to model agents’ participation in stock markets based on available information about risky assets [Antoniou et al. 2015], patients’ decisions in choosing treatment options or participating in medical trials [Attema et al. 2018], and investment decisions in real options [Lourens 2013]. (See [Machina and Siniscalchi 2014] for a survey.)
monotonicity of signals can be defined, we show that both of these properties are satisfied by Receiver’s optimal signal when she is a pessimist. In addition to these structural properties, we also show that the optimal signal is robust to (i) misspecification of the prior and (ii) misspecification of Sender or Receiver’s cardinal utilities as long as their ordinal preferences over the outcomes do not change. We also show that Sender needs a much weaker type of commitment power than the standard one in Bayesian persuasion.

We establish these findings through the lens of an ordinal utility model. While we will provide natural counterparts for the findings in a cardinal model as well, the ordinal model is suitable for capturing the robustness properties of Sender’s signal.

Receiver has a preference list over all outcomes, i.e. pairs of actions and states. After receiving the signal realization, she evaluates any action by the least preferred outcome that it can promise and takes the most promising action. Sender has a preference list over the outcomes and evaluates signals by the probability distribution that they impose on the realized outcome. She chooses a stochastically dominant signal, when it exists, i.e. a signal that imposes a distribution over outcomes which stochastically dominates the distribution imposed by any other signal (where the stochastic dominance relation is defined in the usual way with respect to Sender’s preferences over the outcomes).

We ask whether stochastically dominant signals satisfy the simple structural properties that were discussed earlier, the interval structure and monotonicity of a signal. Merely defining these structural properties requires the action and the state space to be ordered. Formally, we refer to the ordering conditions as action-monotonicity and state-monotonicity. Action-monotonicity means that there exists an ordering on the set of actions such that Sender prefers any higher action in that order to any lower action, at any state. State-monotonicity means that there exists an ordering on the set of states such that Receiver prefers any higher state in that order to any lower state, when taking any action. The two structural properties can be defined only when the action- and state-monotonicity conditions hold. Our main finding is that, then, stochastically dominant signals exist and satisfy both of the structural properties (Section 2). In other words, we find that both of the structural properties hold when both of can be defined.

In “many environments that fit into the persuasion framework, the action space and state space are ordered” [Mensch 2018], and the monotonicity conditions hold. For example, consider a pharmaceutical company persuading customers to buy its product by revealing information about its efficacy, as in [Kamenica and Gentzkow 2017], or a principal persuading an agent to exert effort by providing information about the project reward [Dworczak and
Nevertheless, in some environments, such as in multidimensional persuasion, Receiver’s preference ordering over the states is not complete. Can a relaxed form of the structural properties—the interval structure and monotonicity of a signal—hold in such environments? We extend prior work by allowing Receiver to have an incomplete preference ordering (e.g., a lattice) over the states. We show that stochastically dominant signals still exist and satisfy similar structural properties as in our main findings. This also makes our framework applicable to multidimensional persuasion problems (Section 3.1).

Remarkably, the structural properties of optimal signals heavily rely on the action-monotonicity condition, but their robustness properties do not (Section 3.2). This is also demonstrated in the context of an example in ride-sharing platforms where, without action-monotonicity, the structural properties fail to hold. Nevertheless, we find that even without action-monotonicity, the stochastically dominant signals exist, and their robustness properties hold, as long as the state space is compact and the prior is atomless. Our solution approach here can also be adapted to solve constrained persuasion problems. For example, we show how to find Sender’s optimal signal in the presence of an upper quota constraint on the number of signal realizations. Such constraints may be present in applications where simplicity of the signal is of the design criteria.

Finally, we analyze the size of stochastically dominant signals (number of the signal realizations) in stochastic problem instances. We show that the signals are typically small under our assumptions in Section 4. The analysis heavily relies on our signal construction method from Section 2.

Our findings provide foundation for the simplicity and robustness of optimal persuasion mechanisms in practice by showing that these properties can emerge from the ambiguity of the prior to Receiver. We show that when the conditions for defining both of the simple structural properties—convexity and monotonicity—hold, the optimal signal satisfies both properties if Receiver is a pessimist. This notion of pessimism can also be interpreted through the \textit{Maxmin Expected Utility} model where Receiver is considered an expected utility maximizer who does not know the prior. This explains how ambiguity of the prior can lead to the simplicity of optimal signals.

The rest of the paper is organized as follows. Section 1.1 contains a few preliminary definitions. Our main model and results appear in Section 2. Section 2.2 highlights the connections between the ordinal utility model and the classic cardinal utility model. Section 3 focuses on the existence of stochastically dominant signals and their properties when the monotonicity conditions are relaxed or dismissed. Section 4 analyzes the expected size of
stochastically dominant signals in stochastic problem instances. Section 5 reviews some of the related literature. All the proofs appear in the appendices.

1.1 Preliminary definitions

Preference relations

For any transitive preference relation $\preceq$ over a set, we use $\prec$, $\sim$ respectively to denote the strict preference and the equivalence preference relations imposed by $\preceq$ over that set. For any two elements $x, y$ such that $x \preceq y$, we say that $y$ is $\preceq$-higher than $x$.

Let $\inf \preceq \{X\}$ denote $x \in X$ such that $x \preceq y$ holds for all $y \in X$; if there is more than one such $x$, choose one arbitrarily. Similarly, let $\sup \preceq \{X\}$ denote $x \in X$ such that $y \preceq x$ holds for all $y \in X$.

Asymptotic notions

We say a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of the order of a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ if $f(x)/g(x)$ approaches a constant (possibly zero) as $x$ approaches infinity.

2 The main model and results

There is finite set of states, $\Omega$, and a finite set of actions, $A$. Any element of $A \times \Omega$ is called an outcome. Sender and Receiver have complete and transitive preference relations over the set of outcomes, respectively denoted by $\preceq_s, \preceq_r$.

The state of the world, $\omega_0$, is drawn from a prior $\mu \in \text{int}(\Delta(\Omega))$, henceforth, the prior. Knowing the prior but not the state of the world, Sender chooses a signal, i.e. an arbitrary partition of $\Omega$, namely $\pi = \{P_1, \ldots, P_n\}$. A signal realization $P \in \pi$ is then sent to Receiver such that $\omega_0 \in P$. Receiver then takes an action $a^*(\pi, P)$ that guarantees the best worst-case

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3The assumption that the prior is an interior point is without loss of generality, as Both Sender and Receiver would ignore the states outside of prior’s support.

4The assumptions that signal realizations partition the state space and are sent deterministically are without loss of generality, due to Receiver’s objective.
realized outcome; i.e., $a^* (\pi, P)$ satisfies

$$\inf \{ (a, \omega) : \omega \in P \} \preceq_r \inf \{ (a^* (\pi), \omega) : \omega \in P \}$$

holds where recall that $\inf \{ x \} \in X$ denotes $x \in X$ such that $x \preceq y$ holds for all $y \in X$. (We remark that one way to interpret Receiver’s choice of action is through the Maxmin Expected Utility model, as we discuss in Section 2.2, by considering Receiver an expected utility maximizer with the prior unknown to her.)

Taking Receiver’s objective into account, any signal $\pi$ chosen by Sender imposes a distribution $\eta_{\pi}$ over the set of realized outcomes. Sender’s objective is choosing a stochastically dominant signal, i.e. a signal $\pi^*$ such that $\eta_{\pi^*}$ stochastically dominates $\eta_{\pi}$ for any other signal $\pi$, in the following sense: Let the set of outcomes be ordered as $o_1, \ldots, o_m$ from Sender’s most favorite to her least favorite. Let $p_i, p_{i}^*$ respectively be the probability assigned to outcome $o_i$ in the distributions $\eta_{\pi}, \eta_{\pi^*}$. We say $\eta_{\pi^*}$ stochastically dominates $\eta_{\pi}$ iff for any positive integer $j < m$ with $o_{j+1} \preceq_s o_j$, $\sum_{k=1}^{j} p_k^* \geq \sum_{k=1}^{j} p_k$.

Do stochastically dominant signals, when they exist, satisfy the simple structural properties—the interval structure and monotonicity of a signal? To answer this question, we first define these structural properties formally, which requires defining the notions of “ordered” action and state spaces.

We say Sender’s preferences are monotone in actions when there exists an ordering $\succeq_A$ over actions such that whenever $a \preceq_A a'$, then $(\omega, a) \preceq_s (\omega, a')$ holds for any $\omega \in \Omega$. Receiver’s preferences are monotone in states when there exists an ordering $\succeq_\Omega$ over the states such that whenever $\omega \preceq_\Omega \omega'$, then $(\omega, a) \preceq_r (\omega', a)$ holds for any $a \in A$. For brevity, we sometimes refer to the former condition as action-monotonicity and the latter condition as state-monotonicity. We say Sender and Receiver have monotone preferences when both of these conditions hold.

We are now ready for the promised definition of the structural properties. A signal is convex if all of its signal realizations are convex with respect to $\succeq_\Omega$. We note that since $\succeq_\Omega$ is a complete preference relation over $\Omega$, here, convexity essentially means that the signal realizations partition the ordered state space into “intervals”. A signal is monotone if it recommends $\succeq_A$-higher actions at $\succeq_\Omega$-higher states.

The two structural properties above—convexity and monotonicity of a signal—can be defined only when the action- and state-monotonicity conditions hold. Our main finding is that, then, stochastically dominant signals exist and satisfy both of the structural properties.

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5 Recall the definition of convexity of a signal realization from Section 1.1.
Theorem 2.1. When Sender and Receiver have monotone preferences, there exists a stochastically dominant signal $\pi^*$. Moreover, $\pi^*$ is a monotone signal, and also is convex with respect to $\preceq_\Omega$.

Therefore, the monotonicity conditions that are necessary for defining the structural properties are also sufficient for the existence of stochastically dominant signals, and also for the stochastically dominant signal to satisfy the structural properties.\(^6\)

We prove the existence of stochastically dominant signals by constructing them. Our constructive approach also reveals the following robustness properties of the stochastically dominant signal found by Theorem 2.1:

(i) **Independence from the prior.** The stochastically dominant signal does not depend on Sender’s prior.

(ii) **No commitment to the signal choice.** The usual commitment power assumption in Bayesian persuasion is that, after the state of the world is realized, Sender does not change her choice of signal. Stochastically dominant signals are different: after the state of the world is realized, even if Sender is allowed to send any signal realization that contains the realized state of the world, she would never (strictly) prefer to send a different realization than the one suggested by the stochastically dominant signal.

We emphasize that in the *No commitment to the signal choice* property it is assumed that Sender does not send false signal realizations. This is a milder assumption than the commitment assumption used in the Bayesian persuasion framework where signal realizations are sent randomly. For example, verifying whether the sent signal realization has been correct (contained the true state of the world) is easier than verifying whether the randomization in sending the signal realization has been done correctly, because the only evidence required for verification in the former case is the sent signal realization itself.

Finally, we remark that convexity of the signal could be counted as a consequence of action-monotonicity: in Section 3.2, we see that the signal may be not convex when action-monotonicity is dismissed. Next, we sketch the proofs for Theorem 2.1 and the robustness properties discussed above.

\(^6\) To be more precise, state-monotonicity is necessary for defining the interval structure and monotonicity of a signal, and action-monotonicity is necessary for defining monotonicity of a signal. Furthermore, without action-monotonicity, the interval structure of optimal signals does not necessarily hold, even when state-monotonicity holds (Section B.1).
2.1 Construction of stochastically dominant signals

We first sketch an algorithm that constructs a stochastically dominant signal. (The formal proof is in the appendix, Section B.) This algorithm will also play a key role later in Section 4 in addressing the question of how large stochastically dominant signals can be. To describe the algorithm, we need one definition: an action \( a \) is *inducible* by a subset of states if there exists a state in that subset such that Receiver’s chosen action at that state is \( a \).

The algorithm keeps track of a set of *selected actions* and a set of *covered states*, both initially empty. At each iteration, it chooses the most preferred unselected action with respect to \( \preceq_A \) which is inducible by the set of uncovered states: namely, action \( a \). It then finds the least preferred state with respect to \( \preceq_\Omega \) that induces \( a \): namely, state \( \omega \). At the end of the iteration, \( a \) is added to the set of selected states and a signal realization is added to \( \pi^* \) that contains all uncovered states \( \omega' \) such that \( \omega \preceq_\Omega \omega' \). The states in this signal realization are also added to the set of covered states. The algorithm reiterates until all states are covered. (See Section B in the appendix for a formal definition.)

One can verify that constructed signal, namely \( \pi^* \), is a stochastically dominant signal through property (ii) mentioned above. To see why this property holds, consider an arbitrary state \( \omega \) and an arbitrary subset of the states \( Q \ni \omega \). Furthermore, suppose \( \omega \) belongs to the signal realization \( P \) in \( \pi^* \). We will show that Sender weakly prefers the action induced by \( P \) to the action induced by \( Q \).

Let \( \delta = \inf \preceq_\Omega \{ Q \} \). First, verify that if \( \delta \) is covered before \( P \) is constructed by the algorithm, so should be \( \omega \): since \( \omega \in Q \), we have \( \delta \preceq_\Omega \omega \), and therefore, \( \omega \) should have been covered when \( \delta \) was, by our construction of the signal realizations. This is a contradiction. Therefore, suppose that at the time of constructing \( P \), \( \delta \) is uncovered. In that case, if the action induced by \( Q \) is strictly preferred by Sender to the action induced by \( P \), the algorithm should have chosen the action induced by \( Q \) instead of the action induced by \( P \), since the action induced by \( Q \) is \( (\Omega \setminus \Psi) \)-inducible as well. This would be a contradiction. Consequently, Sender weakly prefers the action induced by \( P \) to the action induced by \( Q \).

Property (i), convexity, and monotonicity of the stochastically dominant signal hold by construction; the complete proof is in the appendix.
2.2 Cardinal utilities and connection to Bayesian persuasion

A natural special case of our model is a cardinal utility model.\footnote{This special case fits into a slightly more general version of the Bayesian persuasion framework of [Kamenica and Gentzkow 2011].} We set up some of our examples in this cardinal utility model, for expositional simplicity. We present the the formal model below for completeness.

A signal $\pi$ consists of a finite realization space $S$ and a family of distributions $\{\pi(\cdot|\omega)\}_{\omega \in \Omega}$ over $S$, i.e. the usual signal structure in Bayesian persuasion. Sender has a prior $\mu$ over $\Omega$, which is shared by Receiver. Sender chooses a signal $\pi$. After the state of the world is realized, a signal realization $s$ is sent to Receiver based on the true state of the world. Then, Receiver forms a posterior $\mu_s$ using the Bayes’s rule and takes the action $a^*(s) = \arg\max_{a \in A} u(a, \mu_s)$, where $u(a, \eta)$ denotes her utility function when she takes action $a$ and her posterior is $\eta$. Sender chooses a signal $\pi$ that maximizes her expected utility, $E_{w \sim \mu} E_{s \sim \pi(\cdot|\omega)} [v(a^*(s), \omega)]$, if such a signal exists. When such a signal exists, it is called the optimal signal.

With a risk-neutral Receiver, i.e. $u(a, \eta) = E_{\omega \sim \eta} [u_0(a, \omega)]$ for some function $u_0 : A \times \Omega \rightarrow \mathbb{R}$, the above model coincides with the Bayesian persuasion model introduced by [Kamenica and Gentzkow 2011].\footnote{It should be pointed out that the main results of [Kamenica and Gentzkow 2011] still hold when the risk-neutrality assumption is dismissed.} For a pessimistic Receiver, we define

$$u(a, \eta) = \inf_{\omega \in \text{supp}(\eta)} u_0(a, \omega),$$

for some function $u_0 : A \times \Omega \rightarrow \mathbb{R}$. We remark that it is also possible to rationalize the utility function $u$ using the Maxmin Expected utility model (as defined, e.g., in [Machina and Siniscalchi 2014]) as the utility function of an expected utility maximizer who does not know the prior.\footnote{After receiving a signal $s$, Receiver considers the set of priors which could generate the received signal, namely the set $\mathcal{F}$. Then, for any prior $\mu \in \mathcal{F}$, she uses the Bayes rule to compute a posterior $\mu_s$ conditional on receiving signal $s$. She then chooses the action $\arg\max_{a \in A} \inf_{\mu \in \mathcal{F}} E_{\omega \sim \mu_s} [u_0(a, \omega)]$. This coincides with the choice of a pessimistic Receiver.}

An optimal signal for sender may not exist in general; Section B.2 in the appendix provides such an example. The existence of the optimal signal, however, is guaranteed under the same monotonicity conditions, action- and state-monotonicity, that we discuss in our main model. When the monotonicity conditions hold, the optimal signal exists, and satisfies the same structural and robustness properties that we discussed in Section 2. Furthermore, the optimal signal is robust to misspecification of Sender or Receiver’s cardinal utilities as...
long as their ordinal preferences over the outcomes do not change. (The latter property is captured by the notion of stochastically dominant signals in our main model.)

Action-monotonicity and state-monotonicity in the cardinal utility model are defined in the natural way: Sender has monotone preferences over the actions when there exists an ordering \( \preceq_A \) over actions such that \( v(a, \omega) < v(a', \omega) \) holds for any state \( \omega \) when \( a \preceq_A a' \). Receiver has monotone preferences over the states when there exists an ordering \( \preceq_\Omega \) over the states such that \( u(a, \omega) < u(a, \omega') \) holds for any action \( a \) when \( \omega \preceq_\Omega \omega' \).

We discuss an example of the cardinal utility model in the appendix (Section B.3) in the context of an ad exchange platform revealing information about the customer’s type to advertisers.

3 Relaxing the monotonicity conditions

Action and state-monotonicity do not hold in some applications. State-monotonicity fails to hold, e.g., in multidimensional persuasion: consider a principal relaying information to an agent about both the “reward” and the “riskiness” of a project. Action-monotonicity might not hold, e.g., when a ride-sharing platform wants to persuade a driver to drive north or south by providing information about the surge factors in each area.

The main question of interest in this section is how relaxing or dismissing one of these conditions can affect the existence and the structural properties of stochastically dominant signals. In Section 3.1 we relax the state-monotonicity condition by allowing Receiver to have a partial preference order over the states, e.g., a lattice. Even though the interval structure of a signal would not be well-defined then, the notion of convexity can be adapted to this relaxation. Similar to our main findings, we see that when the conditions required for defining both of the structural properties—convexity and monotonicity—hold, stochastically dominant signals exist and satisfy both properties.

In Section 3.2, we show that the structural properties of optimal signals heavily rely on the action-monotonicity condition: we demonstrate that, without action-monotonicity, the structural properties fail to hold. Nevertheless, we find that even without action-monotonicity, the stochastically dominant signals exist, and their robustness properties hold, as long as the state space is compact and the prior is atomless.
3.1 Partially ordered state space

Recall from Section 2 that state-monotonicity requires the existence of a complete order $\preceq_\Omega$ over the states. Here, we allow $\preceq_\Omega$ to be a partial order and relax the state-monotonicity assumption to commutativity, which turns out the be equivalent to state-monotonicity when $\preceq_\Omega$ is a complete order. We will provide a counterpart to Theorem 2.1. The proof is by constructing a stochastically dominant signal. The signal will be monotone, convex with respect to Receiver’s partial order, and will also satisfy Independence from the prior and No commitment to the signal choice properties, which were discussed earlier in Section 2.

In the rest of this section, we use $\preceq_\Omega$ to denote a partial order over $\Omega$ which is a lower semilattice.\(^{10}\) For any subset $P \subseteq \Omega$, let $\bigwedge P$ denote the greatest lower bound for $P$ with respect to $\preceq_\Omega$. We say Receiver has commutative preferences when, for any signal realization $P$, Receiver’s optimal choice is determined by $\bigwedge P$, in the sense that $a^*(P) = a^*(\bigwedge P)$. Note that state-monotonicity implies commutativity.

One of the simplest examples of commutative preferences is when $\Omega = \prod_{i=1}^{|A|} \Omega_i$ provides information about Receiver’s actions, with $\Omega_i$ providing information about action $i \in A$. For example, consider a principal-agent problem where Receiver is an agent whose actions correspond to working on one of the $n$ available projects, with $\Omega_i$ providing information about project $i$, e.g., a “risk” or “quality” index in which Receiver’s utility is monotone. Receiver’s preferences then satisfy commutativity but not state-monotonicity.\(^{11}\)

The choice function of a rationally bounded agent with non-commutative preferences also may satisfy commutativity. When solving the max-min optimization problem (i.e. maximizing over actions, minimizing over the signal realization $P$) is difficult for Receiver, choosing an action that maximizes her utility at state $\bigwedge P$ provides her with a guaranteed level of utility through a cognitively simpler task.

The definitions of convexity and monotonicity of a signal from Section 2 extend to this section in a natural way. We include the definitions, for completeness. A signal is convex if all of its signal realizations are convex with respect to $\preceq_\Omega$. A signal is monotone if it recommends $\preceq_\Delta$-higher actions at $\preceq_\Omega$-higher states.

**Theorem 3.1.** When Receiver’s preferences are commutative and Sender’s preferences are monotone in actions, there exists a stochastically dominant signal, which is also a monotone

\(^{10}\)Recall that a lower semilattice is a partial ordering in which any non-empty finite subset of its elements have a greatest lower bound. In particular, any lattice is a lower semilattice.

\(^{11}\)Observe that for any action $i$ and any subset of states $S$, $\min_{\omega \in S} u(i, \omega) = u(i, \min_{\omega \in S}(\omega))$, where the min operator on the right-hand side is the component-wise minimum, taken across different dimensions (i.e. different “indices”, such as “quality” or “risk” index). This implies commutativity.
signal and convex with respect to $\leq_\Omega$.

The stochastically dominant signal still satisfies Independence from the prior and No commitment to the signal choice properties from Section 2. The intuition remains the same as before. In Section 4, we also analyze the expected size of the signal for a special class of commutative preferences, product orders, which also contain the above principal-agent example. We show that the signal is small under the assumptions of that section.

The proof of Theorem 3.1 follows a signal construction approach similar to the one we discussed for Theorem 2.1: At each step, the most preferred unselected action inducible by the uncovered states is selected, and then a signal realization is constructed that contains all the states weakly preferred to that state by $\leq_\Omega$. The states in the signal realization are then added to the set of covered states and the process is repeated until all states are covered.

3.2 Unordered action space

In this section, we show that while the structural properties of stochastically dominant signals heavily rely on the action-monotonicity assumption, their existence and robustness properties do not. We start this section with a new signal construction approach which, in the absence of action-monotonicity, produces an “almost stochastically dominant” signal when the probability that the prior assigns to any “small” subset of the states is “small”. In particular, our construction turns out to be a stochastically dominant when the state space is compact and the prior is atomless. We also adapt our approach here to solve persuasion problems in the presence of upper quota constraints on the signal size, which may be imposed to ensure simplicity. Finally, we provide some insight on this new approach by looking at an example in the context of ride-sharing platforms, where action-monotonicity fails to hold.

3.2.1 The greedy signal

We provide a simple greedy approach that constructs a signal for Sender when Receiver’s preferences are commutative. (Recall the definition of commutativity from Section 3.1.) This signal, henceforth the greedy signal, is constructed by assigning each state of the world to one of the potential signal realizations. The formal construction is given below.
**Algorithm:** Greedy construction of the signal

1. Initialize $\pi, R$ to the empty set.
2. For any action $a$, let $\Omega_a$ be the subset of states that induce $a$.
3. For any action $a$ with $\Omega_a \neq \emptyset$, define $\omega_a = \bigwedge \Omega_a$, add $\omega_a$ to $R$, and let $P_a = \{\omega_a\}$.
4. For any state $\omega \not\in R$, define $a_\omega$ by letting $(a_\omega, \omega) = \sup_{s \preceq s} \{(a, \omega) : a \in A, \omega_a \leq \Omega \omega\}$.
5. For any action $a$ with $\Omega_a \neq \emptyset$, add $\{\omega : a = a_\omega\}$ to $P_a$.
6. Define the greedy signal as $\pi^G = \bigcup_{a \in A : P_a \neq \emptyset} \{P_a\}$.

The algorithm keeps track of a set of representative states, $R$. At most one representative state is added to $R$ for each action $a$, namely $\omega_a$. The algorithm constructs a signal that contains a signal realization per representative state, as follows. Each state $\omega \not\in R$ is greedily assigned to one of the representative states: to the state $\omega_a$ corresponding to the action $a$, where $a$ is Sender’s most preferred action at state $\omega$ among all actions $a'$ that satisfy $\omega_a \leq \Omega \omega$. (Line 4 of the algorithm.) Finally, each representative state together with the states assigned to it are defined as a signal realization in the greedy signal, $\pi^G$.

The key point is that, under the greedy signal, at any non-representative state, Sender takes her most preferred action that could be taken at that state under any signal. This point is a consequence of commutativity: Fix a non-representative state $\omega$ and let $a$ be an action satisfying the constraint $\omega_a \leq \Omega \omega$ in the maximization problem in line 4 of the algorithm. Consider any signal realization containing $\omega$ and suppose that the realization induces action $b$. Then, $\omega_b \leq \Omega \omega$ must hold, by the definition of $\omega_b$ and by commutativity. This means that $(b, \omega)$ appears in the argument of $\sup_{s \preceq s}$ in line 4 of the algorithm, which proves the claim.

This point does not necessarily hold for representative states, and that is why the algorithm does not always construct a stochastically dominant signal. However, if the probability measure that the prior assigns to the set of representative states is “negligible”, then $\pi^G$ is “almost” stochastically dominant. In particular, when the state space is compact and the prior is atomless, $\pi^G$ turns out to be stochastically dominant, as we will formalize next.\(^{12}\)

\(^{12}\)We remark that in the presence of action-monotonicity, the greedy algorithm constructs a stochastically dominant signal even when the state space is discrete. The greedy construction, however, reveals less about the structural properties of stochastically dominant signals (such as their convexity, or, as we will see in Section 4, their size) than the construction approach that we used in Section 2.
3.2.2 The greedy signal in compact state spaces

When the state space is compact, there are pathological partitions of the state space that are unlikely to be of practical interest (e.g. a member of the partition being a fat contour set). To rule out some of the pathological cases, we need a definition.

A signal $\pi$ is proper if the upper contour set (with respect to the order $\preceq_s$) of any outcome is $\eta_\pi$-measurable.\textsuperscript{13} Properness ensures that, given $\pi$ and any outcome $o$, the probability that Sender prefers the realized outcome under $\pi$ to the outcome $o$ is well-defined.

Properness allows the stochastic dominance relation between signals be defined in the natural way when the state space is compact: a proper signal $\pi$ stochastically dominates a proper signal $\pi'$ if for any outcome $p$,

$$\int_{p \preceq s q} 1 \, d \eta_\pi(q) \geq \int_{p \preceq s q} 1 \, d \eta_{\pi'}(q).$$

A proper signal is then called stochastically dominant if it stochastically dominates any other proper signal.

\textbf{Proposition 3.2.} When the prior is atomless, if the greedy signal is proper, it is also stochastically dominant.

By construction, the stochastically dominant signal satisfies \textit{Independence from the prior} and \textit{No commitment to the signal choice} properties that we saw in Section 2. Unlike there, the signal is not necessary convex, as we will see in the ride-sharing example of Section 3.2.3. Therefore, convexity of the stochastically dominant signal can be seen as a consequential feature of action-monotonicity.

In a cardinal utility model, our definition of properness of a signal boils down to a simple one: integrability of Sender’s utility function with respect to the prior, under that signal. The details are discussed in the appendix, Section C.3. There, we also discuss some of the mild sufficient conditions that ensure the properness of the greedy signal. The example of ride-sharing platforms in Section 3.2.3 provides some insight.

Our approach here can be adapted to solve persuasion problems in the presence of exogenous constraints. For example, suppose there is an upper quota constraint on the size of Sender’s signal, i.e. the number of signal realizations. Such constraints could be present to ensure simplicity. The stochastically dominant signal, when it exists, can be found by constructing a series of signals: for each subset of the representative states of the allowed

\textsuperscript{13}Recall the definition of $\eta_\pi$ from Section 2
size, namely $S \subseteq R$, consider a signal with $|S|$ signal realizations. Initially, each signal realization contains precisely one of the states in $S$. Each other state is then added to one of the $|S|$ signal realizations; specifically, to the one that Sender prefers the most.\footnote{Formally, “Sender prefers the most” means the following. Fix the $|S|$ signal realizations and let $\mathcal{P}$ denote the set containing all of them. Consider a state $\omega$ which is supposed to be added to one of the elements of $\mathcal{P}$. Each element of $\mathcal{P}$, namely $P$, induces an action known to Sender, namely $a_P$. The most preferred signal realization (to which $\omega$ will be added) for Sender is then $Q$, where $(a_Q, \omega) = \sup_{P \in \mathcal{P}} \{ (a_P, \omega) : P \in \mathcal{P} \}$.} It can be shown that if one of the produced signals stochastically dominates the other produced signals, then, that signal is also a stochastically dominant signal. In a cardinal utility model, Sender’s optimal signal, which always exists, is her most preferred produced signal.

### 3.2.3 The ride-sharing example

Ride-sharing platforms relay information to drivers about the excess demand, or the “surge multiplier”, in order to control the flow of drivers to different areas. This has been practiced by using illustrative \textit{heat maps} (that may or may not vary across drivers) [Campbell 2018]. The platform’s preferences might not satisfy action-monotonicity: e.g., the platform may want to persuade drivers to go to North only when the surge factor is higher than in South, and vice versa.

Formally, suppose there are two areas, indexed by 0, 1, located at the two extremes of the unit interval. Let $\Omega = \Omega_0 \times \Omega_1$, with $\Omega_i = [\omega_i, \bar{\omega}_i]$ for $i \in \{0, 1\}$, where $\omega_i \in \Omega_i$ denotes the \textit{surge multiplier} at area $i$. For simplicity, we suppose that Driver will surely find a ride if she drives to either of the areas, and that her income from the ride is a dollar times the surge multiplier in that area.

Driver is located at some point on the unit interval, and she has to take an action $a \in \{0, 1\}$, corresponding to driving to area 0 or 1, respectively. Let $c(a)$ denote the cost of taking action $a$. Driver’s payoff from action $a$ is then defined by

$$u(a, \omega) = \beta \omega_a - c(a),$$

where $1 - \beta \in (0, 1)$ is the fraction cut by the platform for commission fee, and $\omega = (\omega_0, \omega_1)$ is the state of the world.

Platform’s payoff function is $v(a, \omega) = (1-\beta)\omega_a$. Therefore, Platform always prefers that Driver drives to the area with the higher surge factor to pick up a ride. Driver’s decision, however, also depends on the cost of each action. What is Platform’s optimal signal, given that its prior over $\Omega$ is $\mu$? As we will see, when Driver is a pessimist, the optimal signal does
not depend on the prior, so long as the prior has full support.

To keep both areas relevant in the solution, assume that \(|c(0) - c(1)| \leq \beta(\omega - \omega')\). (Otherwise, one of the actions will never be taken by Driver, regardless of the state of the world.) Without loss of generality, suppose \(c(0) > c(1)\), and define \(\Delta = (c(0) - c(1))/\beta\). Sender’s optimal signal is illustrated in Figure 1. There are two signal realizations: one corresponding to the shaded area, and the other one to the rest of the state space. When the state of the world falls in the shaded area, Driver receives a message asserting that the surge multiplier in area 0 is above a predetermined fixed threshold, and also larger than the multiplier in area 1. We explain why this signal is optimal next.

Figure 1: The horizontal and vertical axes respectively correspond to \(\Omega_0\) and \(\Omega_1\). From left to right: Receiver’s action at each state under Sender’s optimal signal, the optimal action for Sender at each state, and the optimal action for Receiver at each state. The shaded area corresponds to action 0.

One can verify that Receiver has commutative preferences since her partial order \(\preceq_\Omega\) over \(\Omega\) is defined by \((x, y) \preceq_\Omega (x', y')\) iff \(x \leq x'\) and \(y \leq y'\). Therefore, the greedy algorithm can construct a signal. The optimality of the greedy signal is guaranteed by Proposition C.2 in the appendix, which is a counterpart to Proposition 3.2 but for the cardinal utility model. When the algorithm is run on this example, it first finds two representative states, namely \(\omega_0 = (\Delta, 0)\) and \(\omega_1 = (0, 0)\), corresponding to actions 0, 1, respectively. Then, each other state \(\omega\) is assigned to one of the representative states: to the state \(\omega_a\) corresponding to the action \(a\) that maximizes \(v(a, \omega)\), subject to the constraint that \(\omega_a \preceq_\Omega \omega\). This partitions the set of states to two subsets, producing the optimal signal for Sender, as illustrated in Figure 1. Observe that the convexity of the optimal signal (with respect to \(\preceq_\Omega\)) does not hold here, unlike in Theorems 2.1 and 3.1 where action-monotonicity holds.
4 Size of the signal

We analyze the size of the stochastically dominant signal in stochastic problem instances. For example, suppose that Sender’s utility depends only on Receiver’s action. Then, if Sender’s payoff from each action is i.i.d. across actions, we show that size of the stochastically dominant signal is at most $1 + \log_2 |\Omega|$. After this, we will show that a similar bound holds under much weaker conditions than the i.i.d condition: when there is (even) a small level of misalignment between Receiver’s and Sender’s preferences.

First, we consider an extreme point and give the necessary and sufficient condition for the stochastically dominant signal to have the largest possible size, $|\Omega|$. A signal of this size is also called the fully revealing signal.

For ease of exposition, in this section we suppose that Receiver has strict preferences over the outcomes. The qualitative insights from the next proposition do not hinge on this assumption, and the rest of the propositions hold identically without this assumption.

**Proposition 4.1.** The fully revealing signal is the unique stochastically dominant signal if and only if $a^*(\omega) \prec_A a^*(\omega')$ when $\omega \prec_r \omega'$.

**Proof.** The proof for the if part is straight-forward. It remains to prove the other side. The proof is by contradiction. Suppose any stochastically dominant signal, namely $\pi^*$, has size $|\Omega|$, and suppose there exists $\omega \prec_r \omega'$ such that $a^*(\omega') \preceq_A a^*(\omega)$. Define the signal $\pi'$ as

$$\pi' = \pi \setminus \{\omega\} \cup \{\omega', \omega\},$$

and observe that either $\eta_{\pi'} = \eta_{\pi}$ (which contradicts the uniqueness) or $\eta_{\pi'}$ stochastically dominates $\eta_{\pi}$. Contradiction. \qed

The sufficient and necessary condition in Proposition 4.1 is that Sender’s preference list over actions is the same as the list of Receiver’s optimal actions at each state when the states are ordered with respect to $\preceq_r$. We will refer to this condition as **Receiver’s choices of actions and Sender’s preferences over actions are aligned**, or briefly, **Receiver’s choices and Sender’s preferences are aligned**.

On one extreme, for the stochastically dominant signal to have the largest possible size, Receiver’s choices and Sender’s preferences should be aligned. The other extreme is when they are fully misaligned, in the sense that $a^*(\omega') \prec_A a^*(\omega)$ when $\omega \prec_r \omega'$. Then, the unique stochastically dominant signal is the uninformative (singleton) signal.
Next, we consider a middle ground where Receiver’s choices and Sender’s preferences are, in a sense, independent. We show that in this case as well, the size of the stochastically dominant signal is quite small. After that, we will show that even small levels of misalignment between Receiver’s choices and Sender’s preferences shrink the size of the stochastically dominant signal significantly.

First, we need a definition. Recall that when Senders’ preferences are monotone in actions, \( \preceq_s \) imposes an order \( \preceq_A \) on the set of actions. When the preference relation of Sender, \( \preceq_s \), is a random variable, then so is \( \preceq_A \).

**Definition 4.2.** Suppose \( \preceq_s \) is drawn from a distribution \( D \) (which may depend on \( \preceq_r \)). We say Sender has random independent preferences over actions under \( D \) when \( \preceq_A \) is a random variable independent of \( \preceq_r \) and is distributed uniformly at random over the set of all permutations over \( A \).

When \( D \) is clearly known from the context, we briefly say Sender has random independent preferences over actions. As an example, consider a cardinal utility model where Sender’s utility only depends on Receiver’s action (and not to the state of the world). If Sender’s payoff from each action is i.i.d. across actions, then Sender has random independent preferences over actions. When Sender has random independent preferences, the stochastically dominant signal, \( \pi^* \), is small, in the following sense.

**Proposition 4.3.** Suppose Sender has random independent preferences over actions under distribution \( D \). Then, \( E_D [||\pi^*||] \leq 1 + \log_2 |\Omega| \), where \( \pi^* \) is the (smallest) stochastically dominant signal.\(^{15}\)

The above proposition requires random independent preferences for Sender. The uniformity assumption on \( \preceq_A \), alone, is a mild assumption; e.g., it can be satisfied by relabeling the actions uniformly at random. However, together with independence of \( \preceq_A \) from \( \preceq_r \), they form a strong assumption. Shortly, we will demonstrate that even when the independence assumption is replaced with (even large levels of) positive (or negative\(^{16}\)) correlation, the stochastically dominant signal would still be small.

The proof of Proposition 4.3 builds on our construction of stochastically dominant signals in Section 2. We show that each time an action is selected by the algorithm, in expectation,

\(^{15}\)We can also show that \( ||\pi^*|| \) is of the order of \( E_D [||\pi^*||] \), with high probability.

\(^{16}\)When there is negative correlation, the size of the stochastically dominant signal tends to be smaller than when there is positive correlation. The intuition is simple: at the opposite extreme to the setting of Proposition 4.1, the action taken by Receiver at her least favorite state is Sender’s most favorite action, and therefore the stochastically dominant signal is the uninformative singleton signal.
half of the yet uncovered states are covered, which implies the claimed bound. To prove this, suppose, by the Principle of Deferred Decisions, that Sender’s preferences are generated while the algorithm is run. Therefore, in the first iteration of the algorithm, one can suppose that Sender chooses her most favorite action among the set of inducible actions uniformly at random. Let this action be called $a$. Note that $a$ coincides with the first action selected by the algorithm (to be induced by a signal realization).

By choosing $a$ uniformly at random, about half of the states will be covered by the algorithm (i.e. about half of the states are added to the first signal realization), in expectation. The reason is that each state $\omega \in \Omega$ induces some action, and the probability that an action (such as $a$) is selected by algorithm in the first iteration is proportional to the number of states inducing it. A careful argument shows that the constructed signal realization has the smallest expected size when each state induces a distinct action. In that case, choosing action $a$ uniformly at random from the set of inducible actions is equivalent to choosing a state $\omega$ uniformly at random from $\Omega$. By our construction, all the states higher than $\omega$ will be added to the first signal realization. The expected number of such states is at least $|\Omega|/2$.

Next, we demonstrate that when the independence assumption is replaced with (even large levels of) correlation, the stochastically dominant signal remains small. In particular, we show that even small misalignments between Receiver’s choices and Sender’s preferences can shrink the size of the stochastically dominant signal significantly. We consider a parameterized discrete choice model. At one extreme value of the parameter, Receiver’s choices and Sender’s preferences are fully aligned (i.e. the setting of Proposition 4.1). At the other extreme value, we have random independent preferences (i.e. the setting of Proposition 4.3).

**The discrete choice model.** Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $A = \{a_1, \ldots, a_n\}$, and suppose Receiver and Sender have monotone preferences. Suppose $\omega_1 \preceq \omega_2 \ldots \preceq \omega_n$. Sender’s preferences over outcomes depend only on actions. Let Sender’s preferences over actions, $\preceq_A$, be a random variable drawn from a discrete choice Logit model with weight $\beta_a$ for any action $a \in A$,\(^{17}\) and let $D$ denote the distribution of $\preceq_A$. For any action $a_i$, let $\beta_{a_i} = \theta \log(i)$ for a positive constant $\theta$. (Hence, the probability that action $a_i$ is Sender’s most favorite action is proportional to $i^\theta$, which indicates that the misalignments between Receiver’s choices and Sender’s preferences vanish with a fast rate as $\theta$ increases.) The case of $\theta = 0$, therefore, corresponds to Sender having random independent preferences over actions, and higher values

\(^{17}\)Therefore, the probability that action $a$ is Sender’s most favorite action in any subset $B \subseteq A$ of actions containing $a$ is $\frac{e^{\beta_a}}{\sum_{b \in B} e^{\beta_b}}$. 

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of $\theta$ correspond to higher alignment between Receiver’s choices and Sender’s preferences. We show that, for any fixed $\theta \geq 0$, the expected size of the stochastically dominant signal, $\pi^*$, is of the order of $\ln |\Omega|$.

**Proposition 4.4.** For any non-negative $\theta$, $\mathbb{E}_D[|\pi^*|] \leq c + \log_{\frac{3+\theta}{1+\theta}} |\Omega|$, where $c$ is a constant depending only on $\theta$.

The larger $\theta$, the larger the above bound on size of the signal would be, but no matter how large $\Omega$, the size of the signal is of the order of $\ln |\Omega|$ for any fixed $\theta$.

In the last part of this section, we show that the small size of the dominating signal does not hinge on the state-monotonicity assumption. We provide a counterpart for Proposition 4.3 for when state-monotonicity is replaced with commutativity. The counterpart focuses on a particular family of commutative preferences.

**Definition 4.5.** A preference relation $\preceq$ over $\Omega = \Pi_{i=1}^k \Omega_i$ is a product order when there exists a family of complete preference relations $\{\preceq^i\}_{i=1}^k$ such that $(\omega_1, \ldots, \omega_k) \preceq (\omega'_1, \ldots, \omega'_k)$ holds iff $\omega_i \preceq^i \omega'_i$ holds for all positive integers $i \leq k$.

In Section 3.1 we saw a simple example of a product order preference relation, where $\Omega = \Pi_{i=1}^{|A|} \Omega_i$ and $\Omega_i$ provides information about action $i \in A$. When Receiver has commutative preferences with respect to a product order and Sender has random independent preferences over actions, the (smallest) stochastically dominant signal, $\pi^*$, is small in the following sense.

**Proposition 4.6.** Let $\Omega = \Pi_{i=1}^k \Omega_i$, $\preceq^i$ be a complete and transitive preference relation over $\Omega_i$, and let $\preceq_{\Omega}$ denote the product order defined over $\Omega$ by $\{\preceq^i\}_{i=1}^k$. Suppose Receiver has commutative preferences with respect to $\preceq_{\Omega}$, and Sender has random independent preferences over actions drawn from a distribution $D$. Then, $\mathbb{E}_D[|\pi^*|] \leq 1 + \log_{\frac{2^k}{2^k-1}} |\Omega|$.

The larger $k$, the larger the above bound on size of the optimal signal would be (suggesting that more complex partial preference relations require more signal realizations), but no matter how large $\Omega$, the size of the optimal signal is of the order of $\ln |\Omega|$ for any fixed $k$.

### 5 Related literature

In terms of structural properties of optimal signals, [Ivanov 2015, Kolotilin 2017, Dworczak and Martini 2018, Mensch 2018] are perhaps among the closest to our work from the large literature on Bayesian persuasion. They provide sufficient, and in some cases necessary,
conditions for the sender’s optimal signal to have an “interval structure” in different settings. ([Mensch 2018] focuses on providing sufficient conditions for monotonicity of a signal, a necessary condition for which turns out to be the “interval structure”.) We discuss these work, among others, below. These findings do not imply ours, as we will see. One common difference is that we do not consider Receiver an expected utility maximizer.

[Dworczak and Martini 2018] study the case where the sender’s ex post utility depends only on the mean of posterior beliefs. They provide a sufficient and, in a sense, necessary condition for the optimal signal to have an interval structure. Their condition is in terms of the shape of the sender’s utility function, \( u \): the function \( u \) needs to be “affine-closed”, which, roughly speaking, means that \( u + q \) has at most one local interior maximum for any affine function \( q \).\(^{18}\)

[Ivanov 2015] studies a setting where the action of an expected utility maximizing receiver depends on the mean of posterior beliefs and the order of this mean in the sequence of possible means. He provides a sufficient condition for the optimal signal to have an interval structure. The condition requires the sender’s interim payoff to be linear in posterior means and actions.

[Kolotilin 2017] considers a model where an expected utility maximizing receiver and sender have private types, and receiver takes a binary decision to “act” or not to. To study the interval structure, it is assumed that the sender’s utility only depends on the receiver’s type, and the receiver’s utility is the difference between the sender and the receiver’s types. Taking a linear programming approach, he provides a sufficient condition for the optimality of “interval revelation schemes”, a type of revelation scheme that pools all the “low” states together, fully reveals all the “medium” states, and pools all the “high” states together. Roughly speaking, the condition requires the receiver’s interim utility function to be convex in certain intervals, and requires a first order approximation of the the receiver’s interim utility function to bound the utility function from above in another interval.

The interval structure of messages also appear in other models of communication, e.g. in the cheap talk game of [Crawford and Sobel 1982].

[Mensch 2018] considers an expected utility maximizing receiver and defines the notion of monotone signals as signals that do not recommend lower actions at higher states. He shows that such signals should partition the state space into intervals. Building on the literature on monotone comparative statics, he provides sufficient conditions for a signal to be monotone. In particular, he shows that supermodularity of the sender’s and receiver’s

\(^{18}\)This is a consequence of their general analysis of the case where the sender’s utility depends only on the mean of posterior beliefs.
utility functions is sufficient when the state space is binary. He also considers the case of a continuum of actions. For that case, the sufficient condition is "quasi-supermodularity" of a certain function: the Gateaux derivative of sender’s payoff in the direction of the change of the conditional distribution.

Unlike the above settings and relevant to ours, [Beauchêne et al. 2019] study ambiguity in persuasion games. In their setting, players are ambiguity averse with maxmin expected utility. Even though there is no prior ambiguity, the sender may still choose to use ambiguous communication devices. They characterize the value of optimal ambiguous persuasion, and find it to be often higher than what is feasible under Bayesian persuasion. Their analysis hence provides some justification for how ambiguity may emerge endogenously in persuasion games.

There has also been some work on the robustness of the optimal signals in persuasion problems (e.g. robustness to lack of commitment power or information). [Best and Quigley 2017] address the problem of lack of commitment power in a cheap talk game with a long-lived sender and short-lived receivers. They show that optimal persuasion can be attained by altering the game. [Hu and Weng 2018] consider an ambiguity-averse sender with limited knowledge about the receiver’s private information and a max-min expected utility function. They show that when the sender faces full ambiguity, full disclosure is optimal, and when she faces vanishing ambiguity, she can do almost as well as when receiver has no private information.

References


Ju Hu and Xi Weng. Robust persuasion of a privately informed receiver. *Working paper*, 2018. 21


Jeffrey Mensch. Monotone persuasion. *working paper*, 2018. 1, 2, 19, 20

A Definitions

We need a few definitions for the proofs. We say a state $\omega$ induces action $a$ if Receiver’s chosen action is $a$ when the signal realization is $\{\omega\}$. We say an action $a$ is $\Psi$-inducible if there exists a state $\psi \in \Psi$ that induces $a$.

For any signal $\pi$, recall that $\eta_{\pi}$ denotes the distribution induced over outcomes by $\pi$. Let $a^*(Q)$ denote the action chosen by Receiver when the signal realization is $Q \subseteq \Omega$. We call $a^*$ the choice function of Receiver. A signal realization $Q$ induces an action $a$ when $a^*(Q) = a$.

B Proofs and examples for Section 2

Proof of Theorem 2.1. We start the proof by constructing $\pi^*$, which is done in Algorithm 1. After that, we show that $\pi^*$ satisfies the promised properties.

<table>
<thead>
<tr>
<th>Algorithm 1: Construction of $\pi^*$</th>
</tr>
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<tbody>
<tr>
<td>1. $\Psi \leftarrow \emptyset$</td>
</tr>
<tr>
<td>2. $A^* \leftarrow \emptyset$</td>
</tr>
<tr>
<td>3. $\pi^* \leftarrow \emptyset$</td>
</tr>
<tr>
<td>4. while $\Psi \neq \Omega$ do</td>
</tr>
<tr>
<td>5. Find the most preferred action with respect to $\preceq_A$ in $A \setminus A^*$ which is $(\Omega \setminus \Psi)$-inducible.</td>
</tr>
<tr>
<td>6. $\omega \leftarrow \inf_{\preceq_A} {\omega' : \omega' \text{ induces } a}$</td>
</tr>
<tr>
<td>7. $A^* \leftarrow A^* \cup {a}$</td>
</tr>
<tr>
<td>8. For any $w' \in \Omega \setminus \Psi$ such that $w \preceq_{\Omega} w'$, add $w'$ to $\Psi$ and to $P$.</td>
</tr>
<tr>
<td>9. $\pi^* \leftarrow \pi^* \cup {P}$</td>
</tr>
<tr>
<td>10. end</td>
</tr>
</tbody>
</table>

Observe that the algorithm must terminate, since the size of $\Omega \setminus \Psi$ is reduced at each step. When the algorithm terminates, it has constructed a partition of the states, i.e. the signal $\pi^*$.

To prove the theorem, we need some definitions. We call $A^*$ the set of selected actions and $\Psi$ the set of covered states. For a state $\omega$, we say Sender $\omega$-prefers signal $\pi$ to $\pi'$ if, when the state of the world is $\omega$, she weakly prefers the outcome induced under signal $\pi$ to the outcome induced under signal $\pi'$.

To prove that $\pi^*$ is a stochastically dominant signal, we will show that for any state $\omega \in \Omega$ and any signal $\pi$, Sender $\omega$-prefers $\pi^*$ to $\pi$. Suppose $P$ is the signal realization in $\pi$ that contains $\omega$. Furthermore, suppose $\omega$ belongs to the signal realization $Q$ in $\pi^*$.  

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Let $a, b$ be the actions induced by $P, Q$, respectively, and suppose $b \preceq_A a$. Also, let $\delta = \inf \preceq_\Omega \{Q\}$. The state $\delta$ should be covered by the algorithm at some point before constructing $P$: because otherwise, at the time of constructing $P$, $\delta$ would be uncovered and $b$ would be a more preferred $(\Omega \setminus \Psi)$-inducible action by Sender than $a$, which would be a contradiction.

But then, if $\delta$ is covered before $P$ is constructed by the algorithm, so should $\omega$: because $\delta \preceq_\Omega \omega$ holds since $\omega \in Q$, and therefore $\omega$ should be covered when $\delta$ is, by line 8 of the algorithm. Contradiction.

The Convexity of $\pi^*$ is guaranteed by construction: the only place that signal realizations are added to $\pi^*$ is line 8 of the algorithm. There, we see that whenever a state is included in a signal realization (is covered), all the uncovered states that are preferred by Receiver to that state will also be added to the same signal realization. This guarantees that the convexity holds.

To see why $\pi^*$ is monotone, observe that the algorithm selects the actions in a decreasing order with respect to $\preceq_A$. That is, every time that line 5 is run, the selected action is lower than the previous one. The selected action is the action induced by the signal realization constructed in line 8. (i.e. the new set covered states). The algorithm covers the states in a decreasing order with respect to $\preceq_\Omega$: every time line 8 is run, any state that is added to the signal realization $P$ in that line is preferred less by Receiver to any state that was added in the previous run of line 8. These two facts together imply that the constructed signal is monotone.

\[ \Box \]

B.1 Non-convexity of optimal signals in the absence of action-monotonicity

The utility functions of Receiver and Sender are defined in Figure 2. Observe that while state-monotonicity holds, action-monotonicity does not. Let the prior be the uniform distribution over the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Observe that under any signal, Sender cannot attain utility higher than 1. We show that there exists a unique signal under which Sender attains utility 1, and that the signal is not convex (i.e. does not satisfy the interval structure).

First, observe that Sender attains utility 1 under the signal $\{\{\omega_1, \omega_3\}, \{\omega_2\}\}$. Suppose there is a signal, namely $\pi$ under which Sender to attains utility 1. That means, when the state of the world is $\omega_3$, $\pi$ induces action $a_3$. This implies that $\pi$ either contains the signal realization $\{\omega_1, \omega_3\}$ or the signal realization $\{\omega_1, \omega_2, \omega_3\}$. Observe that Sender attains utility less than 1 in the latter case. Therefore, $\pi = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}$. 

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Figure 2: The left and right tables respectively define the utility functions of Receiver and Sender, \( u, v : A \times \Omega \to \mathbb{R}_+ \).

B.2 Existence of optimal signal in the cardinal utility model

We show that under the cardinal utility model of Section 2.2, an optimal signal does not always exist. In this example, the action- or state-monotonicity assumptions are not satisfied.

Set \( \epsilon \) to be a sufficiently small number; e.g., let \( \epsilon = 0.1 \). The utility functions of Receiver and Sender are defined in Figure 3. Also, the prior is the uniform distribution over the state space \( \Omega = \{\omega_1, \omega_2, \omega_3\} \). We will show that for any sufficiently small \( \delta > 0 \), there is a signal that gives Sender utility \( \frac{2 + 2\epsilon}{3} - \delta \), but there is no signal that gives Sender utility \( \bar{u} = \frac{2(1+\epsilon)}{3} \).

Figure 3: The left and right tables respectively define the utility functions of Receiver and Sender, \( u, v : A \times \Omega \to \mathbb{R}_+ \).

To this end, observe that Sender attains the highest utility at states \( \omega_1, \omega_2, \omega_3 \) respectively under actions \( a_3, a_4, a_2 \). In particular, Sender attains utility \( \bar{u} \) iff Receiver takes actions \( a_3, a_4, a_2 \) at states \( \omega_1, \omega_2, \omega_3 \), respectively. Receiver takes action \( a_3 \) at state \( \omega_1 \) iff the support of her posterior is equal to \( \{\omega_1, \omega_2\} \). But that implies there is a positive probability that she does not take action \( a_4 \) at state \( \omega_2 \). Therefore, there is no signal that gives her utility \( \bar{u} \).

It remains to show that for any sufficiently small \( \delta > 0 \), there is a signal that gives Sender utility at least \( \frac{2 + 2\epsilon}{3} - \delta \). Define the signal \( \pi_\delta \) as follows. The signal space is equal to the state space, i.e. at each state, the signal sends a possibly random state (i.e. signal realization) to Receiver. At state \( \omega_1 \), a signal realization \( \omega_1 \) is sent to Receiver with probability 1. At state \( \omega_3 \), a signal realization \( \omega_3 \) is sent to Receiver with probability 1. At state \( \omega_2 \), a signal realization \( \omega_2 \) is sent to Receiver with probability \( 1 - \delta \), a signal realization \( \omega_1 \) is sent to Receiver with probability \( \delta/2 \), and a signal realization \( \omega_3 \) is sent to Receiver with probability
It is straightforward to verify that under this signal, Receiver attains utility

\[ \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (1 - \delta)(2\epsilon) + \frac{1}{3} \cdot 1 \geq \frac{2 + 2\epsilon}{3} - \delta = u - \delta. \]

This proves the promised claim.

### B.3 Example

First, we present the example assuming that the state space is compact. Later we will also construct the optimal signal for a discrete state space in a similar way.

An ad exchange (Sender) reveals information about a potential customer’s type to an advertiser (Receiver). Type of the customer is a positive number \( \theta \) belonging to the unit interval. After receiving the information, the advertiser posts a bid in an auction to win the opportunity of displaying her advertisement to the customer. In case of winning the auction, the advertiser pays her bid only if the customer clicks on the advertisement.

Suppose that the chance of the event that the advertiser wins the auction and the customer clicks on her advertisement is given by \( q(b, \theta) \), where \( b \) is the advertiser’s bid and \( \theta \) is the customer’s type.

With a customer of type \( \theta \), utility of an advertiser bidding \( b \) is given by

\[ u(b, \theta) = q(b, \theta) \cdot (v(\theta) - b), \]

where \( v(\theta) \) is the expected value generated for the advertiser if a customer of type \( \theta \) clicks on her advertisement. We suppose that \( u(b, \theta) \) is increasing in \( \theta \).

Given a customer of type \( \theta \), define \( b^*(\theta) \) to be the bid that maximizes the advertiser’s utility. (Note that the optimal bid, \( b^*(\theta) \), may be increasing or decreasing in \( \theta \). Also, \( q(b, \theta) \) may be increasing or decreasing in \( \theta \). For example, when the customer’s type is “higher”, there may be more competitor bids which could decrease the chance of winning the auction, holding the bid fixed.)

Advertiser’s utility is given by the function \( R(b, \theta; b_-) \), where \( b \) is the advertiser’s bid, and the vector \( b_- \) denotes her competitors bids. For brevity, we denote this utility function by \( R(b, \theta) \). Suppose that \( R(b, \theta) \) is increasing in \( b \), the advertiser’s bid.

What is Sender’s optimal signal, given that the advertiser’s objective is defined as in Section 2.2? The solution is explained by Figure 4, which illustrates the optimal signal for a given function \( b^* \). Optimality of the solution can be verified using the greedy algorithm.
of Section 3.2. A more intuitive approach is (discretizing the state space and) using the algorithm discussed in Section 2 (i.e. Algorithm 1 in the appendix), as done next.

Suppose \( \theta \) takes discrete equidistance values in the unit interval. (Figure 5) In the first iteration of the algorithm, Sender’s most preferred inducible action (bid) is \( \sup_{\theta} b^*(\theta) \). The state inducing that action is \( \theta = 1 \). Therefore the signal realization constructed by the algorithm is a singleton containing only that state. Similarly, for any \( \theta > \theta_1 \) in the support of \( \theta \), there will be a signal realization containing only that state. When all such states are covered by the algorithm, the highest inducible action would be \( b(\theta_1) = b(\theta_2) \). The lowest state inducing that action is \( \theta_2 \). Therefore, the next signal realization would contain \( \theta_2 \) and all the higher uncovered states, i.e. the states in the support of \( \theta \) belonging to the interval \([\theta_2, \theta_1]\). Construction of the signal continues similarly until all states are covered.

Figure 4: The optimal signal uses pooling in the intervals \([\theta_4, \theta_3]\) and \([\theta_2, \theta_1]\), and uses full revelation everywhere else.

C Proofs from Section 3

C.1 Proofs from Section 3.1

Proof of Theorem 3.1. The proof takes an approach similar to the proof of Theorem 2.1. In particular, it constructs \( \pi^* \) by an algorithm similar to Algorithm 1. First, we recall a few definition. A state \( \omega \) induces action \( a \) if Receiver’s chosen action is \( a \) when the signal realization is \( \{\omega\} \). An action \( a \) is \( \Psi \)-inducible if there exists a state \( \psi \in \Psi \) that induces \( a \). We are now ready for the algorithm that constructs \( \pi^* \), Algorithm 2.
Algorithm 2: Construction of $\pi^*$

1. $\Psi \leftarrow \emptyset$
2. $A^* \leftarrow \emptyset$
3. $\pi^* \leftarrow \emptyset$
4. while $\Psi \neq \Omega$ do
5.  Find the most preferred action with respect to $\preceq_A$ in $A \setminus A^*$ which is $\Omega \setminus \Psi$-inducible; call it action $a$.
6.  $\omega \leftarrow \bigwedge \{\omega' : \omega' \text{ induces } a\}$
7.  $A^* \leftarrow A^* \cup \{a\}$
8.  For any $w' \in \Omega \setminus \Psi$ such that $\omega \preceq \Omega w'$, add $w'$ to $\Psi$ and to $P$.
9.  $\pi^* \leftarrow \pi^* \cup \{P\}$
10. end

The algorithm must terminate, since the size of $\Omega \setminus \Psi$ is reduced at each step. When the algorithm terminates, it has constructed a partition of the states, i.e. the signal $\pi^*$. The proof that $\pi^*$ is stochastically dominant is quite similar to the proof of Theorem 2.1. We include the full proof for completeness. First, we need a few definitions.

We call $A^*$ the set of selected actions and $\Psi$ the set of covered states. For a state $\omega$, we say Sender $\omega$-prefers signal $\pi$ to $\pi'$ if, when the state of the world is $\omega$, she weakly prefers the outcome induced under signal $\pi$ to the outcome induced under signal $\pi'$.

To prove that $\pi^*$ is a stochastically dominant signal, we will show that for any state $\omega \in \Omega$ and any signal $\pi$, Sender $\omega$-prefers $\pi^*$ to $\pi$. Suppose $P$ is the signal realization in $\pi$ that contains $\omega$. Furthermore, suppose $\omega$ belongs to the signal realization $Q$ in $\pi^*$.

Figure 5: The same example with a discrete state space. (The $\theta$-axis is discretized with equidistance dots.)
Let $a, b$ be the actions induced by $P, Q$, respectively, and suppose $b \preceq_A a$. Also, let $\delta = \bigwedge Q$. The state $\delta$ should be covered by the algorithm at some point before constructing $P$: because otherwise, at the time of constructing $P$, $\delta$ would be uncovered and $b$ would be a more preferred $(\Omega \setminus \Psi)$-inducible action by Sender than $a$, which would be a contradiction.

But then, if $\delta$ is covered before $P$ is constructed by the algorithm, so should $\omega$: because $\delta \leq_\Omega \omega$ holds since $\omega \in Q$ and $\delta = \bigwedge Q$, and therefore $\omega$ should be covered when $\delta$ is, by line 8 of the algorithm. Contradiction.

The Convexity of $\pi^*$ is guaranteed by construction: the only place that signal realizations are added to $\pi^*$ is line 8 of Algorithm 2. There, we see that whenever a state is included in a signal realization (is covered), all the uncovered states that are in its upper contour set with respect to $\leq_\Omega$ are also be added to the same signal realization. This guarantees that any signal realization in $\pi^*$ is convex with respect to $\leq_\Omega$.

To see why $\pi^*$ is monotone, observe that the algorithm selects the actions in a decreasing order with respect to $\preceq_A$. That is, every time that line 5 is run, the selected action is lower than the previous one. The selected action is the action induced by the signal realization constructed in line 8. The algorithm covers the states in a non-increasing order with respect to $\preceq_\Omega$ (the order is not necessarily decreasing, since $\leq_\Omega$ may be incomplete). Every time line 8 is run, a state of the world and the uncovered states in its upper contour set with respect to $\leq_\Omega$ are covered (and form a signal realization). Thereby, a state that is covered in later iterations of line 8 cannot be higher (with respect to $\leq_\Omega$) than a state that is covered sooner. This fact together with the fact that the algorithm selects the actions in a decreasing order completes the proof.

\section*{C.2 Proofs from Section 3.2}

\textit{Proof of Proposition 3.2.} Let $\pi_G$ denote the greedy signal.

For a state $\omega$, we say Sender $\omega$-prefers signal $\pi$ to $\pi'$ if when the state of the world is $\omega$, she weakly prefers the outcome induced under signal $\pi$ to the outcome induced under signal $\pi'$.

\textbf{Claim C.1.} For any proper signal $\pi$, Sender $\omega$-prefers $\pi_G$ to $\pi$ for all but a $\mu$-measure zero of the states $\omega \in \Omega$.

If this claim holds, the proof will be complete: suppose not; then there exist a signal $\pi$
and an outcome $p$ such that
\[ \int_{p \leq q} 1 \, d \eta_{\pi_G}(q) < \int_{p \leq q} 1 \, d \eta_{\pi}(q). \]

But that means there exist a positive $\mu$-measure of states such as $\omega$ such that Sender $\omega$-prefers $\pi$ to $\pi_G$, which would be a contradiction. Therefore, it remains to prove Claim C.1.

**Proof of Claim C.1.** We will prove that Sender $\omega$-prefers $\pi_G$ to $\pi$ for any non-representative state $\omega$ (i.e. any $w \not\in R$). Since there are only a finite number of representative states, this will prove the claim.

Suppose $\omega$ is a non-representative state. Therefore in line 4 of the greedy algorithm, $a_\omega$ is defined and in line 5, $\omega$ is assigned to the signal realization represented by $a_\omega$. To prove the claim, we will show that there is no signal realization of $\pi$, namely $P \subseteq \Omega$, such that $(a_\omega, \omega) \prec_s (a^*(P), \omega)$ and $\omega \in P$. For contradiction, suppose there is. Let $b = a^*(P)$. By commutativity, $\Omega_b$ contains $\bigwedge P$, and therefore it is not empty; so, $\omega_b = \bigwedge \Omega_b$ is well-defined. Moreover, $\omega_b \leq_\Omega \omega$ holds, since $\Omega_b$ contains $\bigwedge P$. Therefore, in line 5, we should have
\[ (b, \omega) \leq_s \sup \leq_s \{(a, \omega) : a \in A, \omega_a \leq_\Omega \omega\}, \]
which implies
\[ (a_\omega, \omega) \prec_s (b, \omega) \leq_s (a_\omega, w). \]
Contradiction.

---

**C.3 Optimality of the greedy signal in cardinal utility models**

We provide a counterpart for Proposition 3.2 in a cardinal utility model. The properness condition will be replaced with *validity*, a weaker condition.

For any signal $\rho$, let $\rho(\omega)$ denote Sender's (expected) utility conditioned on the state of the world being equal to $\omega$. A signal $\rho$ is *valid* if $\int_{\omega \in \Omega} v(\rho(\omega), \omega) \, d \mu(\omega)$ is well-defined, i.e. the Lebesgue integral exists.

**Proposition C.2.** Suppose $\Omega$ is compact and $\mu$ is atomless. Then, when the signal constructed by the greedy algorithm is valid, it is also optimal.
Proof. For any valid signal \( \rho \), let \( \rho(\omega) \) denote Sender’s (expected) payoff conditioned on the state of the world being equal to \( \omega \). (The expectation applies in case Receiver uses randomization) We will show that for any signal \( \rho \),

\[
\pi(\omega) \geq \rho(\omega), \quad \forall \omega \not\in R
\]  

(C.1)

holds. Given this inequality, the proof would be complete: Sender’s payoff under the signal \( \pi \) is just equal to \( \int_{\omega \in \Omega} \pi(\omega) \, d\mu(\omega) \). Since there is only a finite number of representative states (with measure 0), and since \( \mu \) is atomless, then by (C.1),

\[
\int_{\omega \in \Omega} \pi(\omega) \, d\mu(\omega) \geq \int_{\omega \in \Omega} \rho(\omega) \, d\mu(\omega)
\]

holds for any signal \( \rho \). Therefore, \( \pi \) is optimal.

It remains to prove (C.1). Recall from the greedy algorithm that, for any action \( a \) with \( \Omega_a \neq \varnothing \), we define \( \omega_a = \bigwedge \Omega_a \). Also, for any state \( \omega \not\in R \), we define

\[
a_\omega = \arg \max_{a \in A: \omega_b = \bigwedge \{ \omega_a, \omega \}} v(a, \omega).
\]

Consider an arbitrary state \( \omega \not\in R \). By construction, \( \omega \in P_{a_\omega} \). Now, consider an arbitrary signal \( \rho \) with \( P \in \rho \) and \( \omega \in P \). Let \( b = a^* (\bigwedge P) \). If we show that \( \omega_b = \bigwedge \{ \omega_b, \omega \} \), then we must have \( v(a_\omega, \omega) \geq v(b, \omega) \), and the proof would be complete. This holds because \( \omega \in P \), \( \bigwedge P \in \Omega_b \), and \( \omega_b = \bigwedge \Omega_b \); hence, \( \omega_b = \bigwedge \{ \omega_b, \omega \} \).

The signals constructed by the greedy algorithm are generally valid for applications of practical interest: Producing signals that are not valid means producing signals that induce non-integrable payoff functions, which is an unlikely case in practical applications. Next, we discuss some of the possible sufficient conditions that ensure the validity of the constructed signal.

C.3.1 Validity of greedy signals

Recall that when the state space is compact, we say a signal \( \rho \) valid when Sender’s expected payoff, \( \int_{\omega \in \Omega} v(\rho(\omega), \omega) \, d\mu(\omega) \), is well-defined, i.e. the latter Lebesgue integral exists. When the integrand has a \( \mu \)-measure zero points of discontinuity (in \( \Omega \)), the existence of the integral is guaranteed. Given that there are only a finite set of actions, this turns out to be satisfied
under mild conditions, as discussed next.

We use two technical conditions to ensure that the the lower semilattice and the payoff functions are “well-behaving”, which in turn guarantees the validity of the constructed signal. After briefing these conditions below, we discuss them more extensively.

**Condition i: on the lower semilattice.** For any \( \omega \in \Omega \), its upper contour set with respect to \( \leq \Omega \), i.e. \( \{ \omega' \in \Omega : \omega \leq \Omega \omega' \} \), is a closed set.\(^{19}\)

**Condition ii: on the payoff functions.** Let a subproblem be defined by a closed subset of the states \( \Psi \subseteq \Omega \) and a subset of the actions \( B \subseteq A \). The condition is that the payoff functions of Sender and Receiver are Lebesgue-integrable under the fully revealing signal in any subproblem. That is, the integrals \( \int_{\omega \in \Psi} u(\rho(\omega), \omega) \, d\mu(\omega) \) and \( \int_{\omega \in \Psi} v(\rho(\omega), \omega) \, d\mu(\omega) \) exist for any closed \( \Psi \subseteq \Omega \) where \( \rho(\omega) \) is Receiver’s optimal action at state \( \omega \) when the set of available actions is limited to \( B \).

For example, when \( \Omega \) is a compact subset of \( \mathbb{R}^n \), the above condition is always satisfied if the functions \( u(\cdot, a), v(\cdot, a) \) are real analytic functions for any action \( a \in A \).

**Proposition C.3.** If Conditions i and ii (defined above) hold, then the greedy signal is valid.

**Proof.** For any representative state \( \omega_a \in R \) corresponding to action \( a \), let \( U_a \) denote the upper contour set of \( \omega_a \) with respect to \( \leq \Omega \). For any subset of actions \( B \subseteq A \) define

\[
\xi(B) = \left( \bigcap_{x \in B} U_x \right) - \left( \bigcup_{y \not\in B} U_y \right).
\]

Observe that any \( \omega \in \Omega \) that could be induced by an action, there exists a unique subset \( B \subseteq A \) that contains \( \omega \). Therefore, to prove that the constructed signal is valid, it suffices to show that the integrals \( \int_{\omega \in \xi(B)} u(\rho_B(\omega), \omega) \, d\mu(\omega) \) and \( \int_{\omega \in \xi(B)} v(\rho_B(\omega), \omega) \, d\mu(\omega) \) exist where \( \rho_B(\omega) \) is Receiver’s optimal action at state \( \omega \) when the set of available actions is limited to \( B \). We give the proof for the existence of the former integral, \( \int_{\omega \in \xi(B)} u(\rho_B(\omega), \omega) \, d\mu(\omega) \). The proof for the existence of the latter integral follows similarly. To prove the existence of the former integral, it suffices to prove that the integrals

\[
\int_{\omega \in \bigcap_{x \in B} U_x} u(\rho_B(\omega), \omega) \, d\mu(\omega)
\]

\(^{19}\)An alternative condition, e.g., is that the boundary of the upper contour set has \( \mu \)-measure zero.
and
\[ \int_{\omega \in (U_{y \notin B} U_y)} u(\rho_B(\omega), \omega) \, d\mu(\omega) \]
exist. Since both \((\bigcap_{x \in B} U_x)\) and \((\bigcup_{y \notin B} U_y)\) are closed sets by Condition i, therefore the existence of the integrals are guaranteed by Condition ii.

We will show that the function \(\rho_B\) is discontinuous only over a \(\mu\)-measure zero set of points in \(\xi(B)\). To this end, let \(f\) be the correspondence
\[ f(x) = \{ a : u(a, x) = \max_{b \in A} u(b, x) \}, \]
defined for any \(x \in B\). We say a point \(x \in \xi(B)\) is stable if there exists an open ball around it such that for any point \(y\) in that ball \(f(x) = f(y)\). To prove the claim, it suffices to show that the set of unstable points in \(\xi(B)\) has a \(\mu\)-measure zero. First, we prove this assuming \(|B| = 2\). The proof for \(|B| > 2\) follows by induction, as we will see later.

**The case of \(|B| = 2\).** Suppose \(B = \{a, b\}\). Define \(g(x) = u(a, x) - u(b, x)\) for all \(x \in \xi(B)\). Observe that any point \(x\) with \(g(x) \neq 0\) is stable, by the continuity of \(u(a, \cdot)\) and \(u(b, \cdot)\). Therefore, to prove the claim, it suffices to show that the set of roots of \(g\) has \(\mu\)-measure zero. This is readily implied from the fact that \(g\) is an analytic function itself, and therefore the set of its roots has Lebesgue-measure zero [Mityagin 2015].

**The case of \(|B| > 2\).** For any two actions \(a, b\), define the function \(g_{a,b}(x) = u(a, x) - u(b, x)\).
Let \(Z_{a,b}\) denote the set of roots of \(g_{a,b}\). By an argument similar to the case of \(|B| = 2\), \(Z_{a,b}\) has \(\mu\)-measure zero. Let \(Z = \cup_{\{a,b|ca\neq b,a\in A\}} Z_{a,b}\). Since \(A\) is finite, \(Z\) also has a \(\mu\)-measure zero. The set of stable points is a subset of \(Z\), and therefore has a \(\mu\)-measure zero as well.  

\[ \square \]

**D Proofs from Section 4**

**Proof of Proposition 4.3.** The proof uses the Principle of Deferred Decisions. Rather than fixing the composition of \(\preceq_s\), we generate it while running Algorithm 1. We first need to make a few definitions. Let \(a^*(Q)\) denote the action chosen by Receiver when the signal realization is \(Q \subseteq \Omega\). We say a signal realization \(Q\) induces an action \(a\) when \(a^*(Q) = a\). When \(Q = \{\omega\}\) is singleton, with slight abuse of notation we also denote \(a^*(Q)\) by \(a^*(\omega)\).
Let \(\hat{A} = \cup_{\omega \in \Omega} a^*(\omega)\), and for any \(a \in \hat{A}\), let \(a^{-1}(a)\) denote the set \(\{\omega : a^*(\omega) = a\}\).
When Algorithm 1 is run, it finds the most preferred action (with respect to $\preceq_\mathcal{A}$) in $\hat{\mathcal{A}}$ that is inducible; this is done in line 5. Consider the first time that line 5 is run. When it comes to choosing the most preferred action, we use the Principle of Deferred Decisions, and rather than defining $\preceq_\mathcal{A}$ over $\hat{\mathcal{A}}$ completely, we only determine $\sup_{\preceq_\mathcal{A}} \{ \hat{\mathcal{A}} \}$. This can be done simply by choosing one of the elements in $\hat{\mathcal{A}}$ uniformly at random, since Sender has random independent preferences over actions. Suppose the chosen element is $a$ (following the notation in Algorithm 1). The algorithm then proceeds to line 8 where the first signal realization, $P$, is constructed. Verify that

$$P = \{ \psi : \inf_{\preceq_\Omega} \{ a^{-1}(a) \} \preceq_\Omega \psi \}$$

We will show that the expected size of $P$ is at least $|\Omega|/2$, and then use this fact to complete the proof by repeatedly applying the same argument.

**Claim D.1.** When $a$ is chosen uniformly at random from $\hat{\mathcal{A}}$, then the expected size of $P$ is at least $|\Omega|/2$.

**Proof.** Define $P' \subseteq \Omega$ as follows. Choose $\omega'$ uniformly at random from $\Omega$. Then, let $P' = \{ \psi : \omega' \preceq_\Omega \psi \}$. It is straightforward to verify that the expected size of $P'$ is at least $|\Omega|/2$. To prove the claim, we will show that the expected size of $P$ is at least equal to the expected size of $P'$. Note that in the construction of $P$, any element of $\Omega$ is a member of $a^{-1}(a)$ with probability at least $1/|\Omega|$. On the other hand, in the construction of $P'$, each element of $\Omega$ is chosen as $\omega'$ with probability $1/|\Omega|$. This fact together with the definitions of $P, P'$ imply that the expected size of $P$ is at least equal to the expected size of $P'$. \qed

We are now ready to finish the proof. Suppose $x = |\Omega|$ and let $T(x)$ denote the expected size of the signal that is constructed by Algorithm 1 when the state space given its input has size $x$. Note that by line 8, $\Psi = P$. After the first iteration of the loop is completed, the rest of the algorithm is run essentially by removing $\Psi$ from $\Omega$ and repeating the same loop. We therefore can write

$$T(x) = 1 + \mathbb{E}_a [T(x - |P|)], \quad (D.1)$$

where the expectation is taken over the choice of $a$. (Recall that $a$ is chosen uniformly at random from $\hat{\mathcal{A}}$.)
In the rest of the proof, we use induction to show that $T(x) \leq 1 + \log_2 x$. The base case for $x = 1$ is trivial. Suppose $x > 1$. We then can write

$$
T(x) \leq 1 + \mathbb{E}_a [1 + \log_2 (x - |P|)],
$$

(D.2)

$$
\leq 2 + \log_2 (\mathbb{E}_a [x - |P|])
$$

(D.3)

$$
\leq 2 + \log_2 (\mathbb{E}_a [x/2]) = 1 + \log_2 x,
$$

(D.4)

where (D.2) holds by (D.1) and the induction hypothesis, (D.3) is by the Jensen inequality, and (D.4) holds by Claim D.1. The proof is complete.

Proof of Proposition 4.6. The proof follows a similar approach as the proof of Proposition 4.3. Let $\hat{A} = \cup_{\omega \in \Omega \setminus \Psi} a^*(\omega)$, where $\Psi$ is defined in Algorithm 2. Also, for any $a \in \hat{A}$, let $a^{-1}(a)$ denote the set $\{\omega : a^*(\omega) = a\}$. When Algorithm 2 is run, it finds the most preferred action (with respect to $\preceq_A$) in $\hat{A}$ that is inducible; this is done in line 5. When it comes to choosing the most preferred action, we use the Principle of Deferred Decisions, and rather than defining $\preceq_A$ over $\hat{A}$ completely, we only determine $\sup_{\preceq_A} \{\hat{A}\}$. This can done simply by choosing one of the elements in $\hat{A}$ uniformly at random, since Sender has random independent preferences over actions. Suppose the chosen element is $a$ (following the notation in Algorithm 2).

The algorithm then proceeds to line 8 where the first signal realization, $P$, is constructed. Verify that

$$
P = \{\psi : \psi \in \Omega \setminus \Psi, \bigwedge a^{-1}(a) \leq_{\Omega} \psi\}.
$$

We will show that the expected size of $P$ is at least $|\Omega \setminus \Psi|/2^k$. We then use this fact to complete the proof by repeatedly applying the same argument.

Claim D.2. When $a$ is chosen uniformly at random from $\hat{A}$, then the expected size of $P$ is at least $|\Omega \setminus \Psi|/2^k$.

Proof. The proof is induction. The induction basis is $k = 1$, which is proved in Claim D.1. For the induction step, suppose $k > 1$.

Define $P' \subseteq \Omega$ as follows. Choose $\omega'$ uniformly at random from $\Omega \setminus \Psi$. Then, let

$$
P' = \{\psi : \psi \in \Omega \setminus \Psi, \omega' \leq_{\Omega} \psi\}.
$$

First, we will show that the expected size of $P$ is at least equal to the expected size of $P'$, and then we will show that the expected size of $P'$ is at least $|\Omega \setminus \Psi|/2^k$. For the first step, note
that in the construction of $P$, any element of $\Omega \setminus \Psi$ is a member of $a^{-1}(a)$ with probability at least $1/|\Omega \setminus \Psi|$. On the other hand, in the construction of $P'$, each element $\omega \in \Omega \setminus \Psi$ is chosen as $\omega'$ with probability $1/|\Omega \setminus \Psi|$. This fact together with the definitions of $P, P'$ imply that the expected size of $P$ is at least equal to the expected size of $P'$.

It remains to show that the expected size of $P'$ is at least $|\Omega \setminus \Psi|/2^k$. To this end, define $S = \{\omega_1 : (\omega_1, \ldots, \omega_k) \in \Omega \setminus \Psi\}$. Also, denote $|S|$ by $m$ and w.l.o.g. suppose that elements of $S$ are $\omega_1^1 \leq \cdots \leq \omega_m^1$. For any positive integer $i \leq m$ define

$$T_i = \{(\omega_1, \ldots, \omega_k) \in \Omega \setminus \Psi : \omega_1 = \omega_i^1\},$$

i.e. the elements in $\Omega \setminus \Psi$ with $\omega_i^1$ as their first component. Let $x_i = |T_i|$.

We now provide a lower bound on the expected size of $P'$. Recall the definition of $P'$,

$$P' = \{\psi : \psi \in \Omega \setminus \Psi, \omega' \leq_\Omega \psi\},$$

where $\omega'$ is chosen uniformly at random from $\Omega \setminus \Psi$. Conditioned on $\omega' \in T_i$, for any $j \leq i$ the expected size of the elements from $T_j$ which will be added to $P'$ is at least $x_i/2^{k-j}$. This follows from the induction hypothesis. This fact implies that the expected size of $P'$ is at least

$$\sum_{i=1}^m (i - m + 1) \cdot \frac{x_i}{|\Omega \setminus \Psi|} \cdot \frac{x_i}{2^{k-1}}.$$

Denote the above quantity by $f(x_1, \ldots, x_m)$. Observe that $f$ is a convex function of $x$ (e.g. by verifying that it has a positive semi-definite Hessian). Therefore, the minimum value of $f$ is attained at a point where $x_1 = \ldots = x_m$, subject to the constraint that $\sum_{i=1}^m x_i = |\Omega \setminus \Psi|$. This implies that $f(x)$ is at least $|\Omega \setminus \Psi|/2^k$, which proves the claim.

We are now ready to finish the proof. Let $x = |\Omega \setminus \Psi|$ and let $T(x)$ denote the expected number of signal realizations that Algorithm 2 adds to $\pi^*$ when the remaining state space, $\Omega \setminus \Psi$, has size $x$. We then can write

$$T(x) = 1 + \mathbb{E}_a \left[T(x - |P|)\right],$$

where the expectation is taken over the choice of $a$. (Recall that $a$ is chosen uniformly at random from $\hat{A}$.)

In the rest of the proof, we use induction to show that $T(x) \leq 1 + \log_{2^{1/k}} x$. The base
case for \( x = 1 \) is trivial. Suppose \( x > 1 \). We then can write

\[
T(x) \leq 1 + \mathbb{E}_a \left[ 1 + \log \frac{2^k x}{2^k - 1} (x - |P|) \right], \tag{D.6}
\]

\[
\leq 2 + \log \frac{2^k}{2^k - 1} \left( \mathbb{E}_a [x - |P|] \right) \tag{D.7}
\]

\[
\leq 2 + \log \frac{2^k}{2^k - 1} \left( \mathbb{E}_a \left[ x \cdot \frac{2^k - 1}{2^k} \right] \right) = 1 + \log \frac{2^k}{2^k - 1} x, \tag{D.8}
\]

where (D.6) holds by (D.5) and the induction hypothesis, (D.7) is by the Jensen inequality, and (D.8) holds by Claim D.2. The proof is complete.

\[
\square
\]

**Proof of Proposition 4.4.** We use the Principle of Deferred Decisions and rather than fixing \( \preceq_A \), we construct it in the course of running Algorithm 1.

To this end, let \( \hat{A} = \cup_{\omega \in \Omega} a^*(\omega) \), and for any \( a \in \hat{A} \), let \( a^{-1}(a) \) denote the set \( \{ \omega : a^*(\omega) = a \} \). Consider the first iteration of the while loop in Algorithm 1: In line 5, the most preferred action (with respect to \( \preceq_A \)) in \( \hat{A} \) that is inducible is found. When it comes to choosing the most preferred action, we use the Principle of Deferred Decisions, and rather than defining \( \preceq_A \) over \( \hat{A} \) completely, we only determine \( \sup_{\preceq_A} \{ \hat{A} \} \). This can done simply by choosing one of the elements in \( \hat{A} \) at random, with each element \( a \) being chosen with probability proportional to \( e^{\beta a} \). Let the chosen element be \( a \) (following the notation in Algorithm 1). The algorithm then proceeds to line 8 where the first signal realization, \( P \), is constructed. Verify that

\[
P = \{ \psi : \inf_{\Omega} \{ a^{-1}(a) \} \preceq_{\Omega} \psi \}.
\]

We will show that the expected size of \( P \) is at least \( |\Omega|/(\theta + 3) \). We then use this fact to complete the proof by repeatedly applying the same argument.

**Claim D.3.** When \( |\Omega| > n_\theta \), the expected size of \( P \) is at least \( |\Omega|/(\theta + 3) \), where \( n_\theta \) is a constant depending only on \( \theta \).

**Proof.** For notational simplicity, let \( n = |\Omega| \). When sampling an action from \( \hat{A} \), action \( a_i \) is chosen with probability proportional to \( i^\theta \), in which case \( |P| = n - i + 1 \). Therefore, we can
write
\[
\mathbb{E}[|P|] = \frac{\sum_{i=1}^{n}(n-i+1)i^\theta}{\sum_{i=1}^{n}i^\theta} = n + 1 - \frac{\sum_{i=1}^{n}i^{\theta+1}}{\sum_{i=1}^{n}i^\theta} \geq n + 1 - \frac{\int_{0}^{n+1}i^{\theta+1}di}{\int_{0}^{n}i^\theta di} = n + 1 - (n + 1) \cdot \frac{\theta + 1}{\theta + 2} \cdot (1 + 1/n)^{\theta+1}.
\]

This implies that for all \( n > n_\theta \), \( \mathbb{E}[|P|] \geq \frac{n}{\theta + 3} \), where
\[
n_\theta = \left( \left( \frac{(\theta + 2)^2}{(\theta + 1)(\theta + 3)} \right)^{1/(\theta + 1)} - 1 \right)^{-1}.
\]

We are now ready to finish the proof. Suppose \( T(n) \) denote the expected size of the signal that is constructed by Algorithm 1 where \( n = |\Omega| \). Note that by line 8, \( \Psi = P \). After the first iteration of the loop is completed, the rest of the algorithm is run essentially by removing \( \Psi \) from \( \Omega \) and repeating the same loop. We therefore can write
\[
T(n) = 1 + \mathbb{E}_a[T(n - |P|)], \quad (D.9)
\]
where the expectation is taken over action \( a \).

In the rest of the proof, we use induction to show that \( T(n) \leq c + \log_{\frac{4+\theta}{2+\theta}} n \), where \( c \) is a constant depending only on \( \theta \). To keep the proof simple, we define \( c = n_\theta \). (E.g., \( n_1 = 15 \). A tighter analysis can reduce the value of \( c \); this is not our focus here.) The base cases for \( n \leq n_\theta \) are trivial. Suppose \( n > n_\theta \). We then can write
\[
T(n) \leq 1 + \mathbb{E}_a\left[ c + \log_{\frac{4+\theta}{2+\theta}} (n - |P|) \right], \quad (D.10)
\]
\[
\leq 1 + c + \log_{\frac{4+\theta}{2+\theta}}(\mathbb{E}_a[n - |P|]) \quad (D.11)
\]
\[
\leq 1 + c + \log_{\frac{4+\theta}{2+\theta}}(\mathbb{E}_a[n(\theta + 2)/(\theta + 3)]) = c + \log_{\frac{4+\theta}{2+\theta}} n, \quad (D.12)
\]
where (D.10) holds by (D.9) and the induction hypothesis, (D.11) is by the Jensen inequality, and (D.12) holds by Claim D.3. The proof is complete.

\[\square\]