Thickness and Competition in On-demand Service Platforms

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Abstract

A distinct feature of *on-demand service platforms* is dependence of their service quality to the size of their labor pool. We show that this property can sharply change the equilibrium characteristics when the market is *thin* (i.e., the labor pool is not sufficiently large). Namely, (i) improving service quality for customers—through increasing the labor pool or improving the *matching technology*—also increases workers wage and average welfare, and (ii) competition between platforms has an adverse effect on customers: price is higher and customers' average welfare is lower in the duopoly equilibrium than in the monopoly equilibrium.

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1 Introduction

On-demand service platforms, such as those offering ride-hailing, courier, or delivery services, realize a mode of the sharing economy whereby the platform mediates between *workers* and potential *customers* whose request for on-demand services can be satisfied only if there are workers willing and available to provide service. Such platforms invest in improving service quality for customers, often by expanding their labor pool and by improving their resource allocation strategies to provide better matches or recommendations. How does improving service quality for customers affect workers? And how does the competition between platforms affect customers and workers? These questions, which concern the platforms as well as the market regulators, are the main subject of this work.

Increasing the size of the labor pool leads to both expected and unexpected implications. By analogy to a classical marketplace, increasing the labor pool (i.e., increasing the number of potential workers) increases supply and makes it cheaper to provide any level of service, so customers' average welfare generally goes up. What is different about on-demand service platforms is that a larger labor pool can lead the firm to offer *higher* wages to workers, so that the workers' average welfare increases, too. The intuition for this reversal is founded in the observation that increasing the size of the labor pool pressures the equilibrium wage in two ways. First, increasing the labor pool increases the number of workers at any wage, raising quality (for example, through shorter wait times for customers) and pushing up the firm's marginal expenditure to improve quality. Second, increasing the labor pool also reduces the firm's marginal expenditure to acquire an additional worker, pushing down the firm's marginal expenditure to improve quality. We show that the second effect in this trade-off can dominate the first one, implying that a larger labor pool can make it profitable for the firm to offer a higher wage.

We study this trade-off and its determinants in Section 4. We show that in *thin* markets (i.e., markets with not a sufficiently large labor pool), the workers' wage, their average employment time, and their average welfare *increase* as the labor pool increases (Theorem 4.1). So, the workers are "complements" in thin markets, whereas they typically become "substitutes" and compete with each other in thick markets.[1] To see the intuition, recall the trade-off discussed above. When the market is thin, service quality is low and marginal increase in quality for an increase in wage is high, tilting the trade-off toward increasing wage.

[1]Which suggests that it can be more difficult for the platform to attract workers in thick markets.

Besides expanding the labor pool, platforms can also improve service quality by improving their resource allocation strategies. We study the effect of improved *matching technology*, i.e., improving the matching algorithm of the firm so that service quality goes up given the same labor supply. We show that improving the matching technology can have an effect similar to that of increasing the labor pool, benefiting workers when the market is thin, while otherwise reducing their wage and average welfare. In other words, matching technology complements labor in thin markets, and substitutes it otherwise. Section 5 studies this effect and its determinants.

In sum, the above findings show that, when the market is thin, improving service quality for customers—through expanding the labor pool or improving the matching technology—also increases workers' average welfare and wage.

We find that, in thin markets, the effect of platform competition also differs from what one might expect. Then, competition has an adverse effect on customers, in the sense that the price is higher and customers' average welfare is lower in the duopoly equilibrium (where two platforms compete) than in the monopoly equilibrium.

The intuition is simple. Two main forces affect the duopoly price. Competition for customers pushes the customer price down, but there is another effect of competition that is adverse to customers: competition for workers raises the firms' costs, pushing the customer price up. The net effect of competition on price depends on the strength of these forces. When the market is thin, competition over workers dominates competition over customers, the price for customers is higher in the duopoly equilibrium than in the monopoly equilibrium, and customers' average welfare is lower. (Section 6)

Finally, we point out to some of the prominent features of on-demand service platforms which are detrimental to the effects that we introduce. A feature related to the effect of competition is that the multihoming side (i.e., workers, who can accept requests from both platforms) has limited availability.[2] For example, workers may not be able to serve multiple customers simultaneously at both firms. This characteristic intensifies the competition for workers, which benefits the multihoming side but has an adverse effect on the singlehoming side (customers) in thin markets (Theorem 6.5). This is in contrast to the results in the classic two-sided platforms literature, where the singlehoming side typically benefits from competition and the surplus of the multihoming side can be fully extracted, as firms do not directly compete for them [Armstrong 2005, Lam 2017].

[2] Customers are the single-homing side as they request service only from one firm (while being able to observe service quality at all firms.

Another feature of on-demand service platforms that plays a crucial role in the effect of improving service quality is convexity of the (waiting) cost of a customer in the number of available idle workers: the marginal reduction in (waiting) cost for an additional idle worker is decreasing in the number of idle workers. Similar convexities could be present in other two-sided marketplaces as well, deriving effects similar to the ones that we introduce here. The magnitude of these effects can depend on the finer marketplace details (which vary from market to market), as discussed in Section 7. There, we also perform several robustness checks in models with finer details and under different assumptions.

2 Related work

In this section we review some of the relevant work in the literature of two-sided platforms, network effects, and dynamic pricing.

The literature on platform competition and two-sided markets offers generic insights on how market equilibria change under different governance structures (e.g., a social welfare or a profit-maximizing planner) and highlights the significance of some key structural components of two-sided markets, such as user heterogeneity and multihoming and singlehoming of users, among others. Some of these works are reviewed below. They consider different comparative statics than we do. Also, they are generally tailored for different market structures, such as credit card markets. Such differences in design details can create significantly different equilibrium properties, as we briefly explained in Section 1.

[Rochet and Tirole 2003, Tirole and Rochet 2006] introduce a generic framework and use it to compare end-user surpluses for different planners, study the determinants of the business model (the favorability of the price structure on each side of the market), and investigate different membership structures.

[Caillaud and Jullien 2003] and [Armstrong 2005] elaborate the role of singlehoming and multihoming users on market equilibria and users' surplus and show that the singlehoming side is treated favorably. [Armstrong and Wright 2007] argue that multi-homing users could result in "competitive bottlenecks" in a market and study exclusive deals to prevent multihoming. More precisely, they show that when platforms are viewed as homogeneous by multihoming users but heterogeneous by singlehoming users, they do not compete directly for multi-homing users, and instead, choose to compete for them indirectly by subsidizing single-homing users to join. In contrast, in this paper we find competition to benefit the multihoming side (workers), while adversely affecting the singlehoming side when the labor pool is not sufficiently large.

[Weyl 2010] highlights the role of user heterogeneity in normative properties and comparative statics of two-sided markets. He reformulates a platform's problem in terms of the allocation choice, rather than prices, while allowing for user heterogeneity in income or scale. [White and Weyl 2010] suggest insulated equilibrium as a novel equilibrium notion and show that under this notion, the impact of competition (defined by the level of product differentiation) on efficiency depends on heterogeneity in users' valuations for network effects.

More recently, [Tan and Zhou 2020] study the effect of competition in multisided platforms in the presence of network externalities and detect conditions under which the equilibrium price can increase with the number of competing platforms. Their findings differ from the adverse effect of competition introduced here in that their conditions are related to the distributional form of the customers' valuations, whereas the main condition in here is related to market thickness.

There is also extensive literature on network externalities and economics of networks; [Shy 2011] gives a brief survey. One generic intuition is that expanding both sides of a platform simultaneously could benefit both sides (compared to the effect that we characterize, where expanding the labor side alone benefits that same side when the labor pool is small). Several works in this area study competition between firms in the presence of network externalities. [Katz and Shapiro 1985] show that firms' joint incentives for product compatibility are lower than the social incentives. [Economides 1996] explains that the existence of network externalities cannot be claimed as a reason in favor of a monopoly market structure, as their presence "does not reverse the standard welfare comparison between monopoly and competition."

[Cournot et al. 1927] show that nonintegrated dual monopolists can quote higher prices than a single vertically integrated monopolist in the pricing of two perfectly complementary goods. The intuition is that dual monopolists face less elastic demand and quote higher prices than a single vertically integrated monopolist (an effect also known as "double-marginalization"). [Economides 1999] extends this result by showing that product quality will also be higher under a single integrated monopolist when the quality choices are endogenous and the goods are perfect complements. We remark that the adverse effect of competition on customers that we introduce arises not because of the complementarity of the goods (in fact, the two goods provided by the two firms are substitutes), but because of the competition over the multihoming side.

The Operations Research literature includes some work on on-demand service platforms. [Taylor 2018] examines how delay sensitivity and agent independence impact a platform's optimal per-service price and wage. In a working draft subsequent to ours, [Benjaafar et al. 2018] study labor welfare in on-demand service platforms in a different model. Some differences are that (i) they make the following functional form assumptions: a customer's valuation is drawn from the uniform distribution, the opportunity cost of a worker is drawn from a distribution with a quadratic PDF, and the waiting time function has a specific functional form; (ii) they allow each worker to choose the amount of her availability according to the utilization rate and wage, and (iii) we micro-fund the availability rate of workers and the externality imposed by serving more customers on the waiting times. They study the welfare impacts of increasing the size of the labor pool and develop results that are aligned with our Theorem 4.1. They also study the welfare impacts of imposing a wage floor.

[Ahmadinejad et al. 2019] study the possibility of market failure under platform competition in ride-hailing markets. They detect conditions under which high-throughput equilibria exist, due to the possibility of rapid deterioration of market throughput deterring the platforms from undercutting each other's prices.

3 Setup

The model is a dynamic steady-state model. A single firm mediates between customers and workers. Potential customers arrive according to a flow with a constant rate that we normalize to 1. There is also a mass m of potential workers, which we call the *labor pool*. The firm makes matches between customers and workers. Immediately upon arrival, a customer either requests service or departs the market. When a customer requests service, the firm matches her to a worker. The firm selects the worker uniformly at random from the set of workers who accept the firm's wage offer and are not busy serving other customers. When the firm makes a match between a customer and a worker, the worker becomes *busy* for a unit of time during which she serves the customer. After that, the customer departs the market. The firm posts a price p and a wage w, which respectively are the payment from a customer to the firm and the payment from the firm to a worker, upon a match.

Next we define the decision problems of the workers, the customers, and the firm.

Workers

Each worker has an outside option r, drawn independently from a CDF F, which represents the *opportunity cost* of the worker: the worker earns r per unit of time whenever she is not serving a customer.[3] The workers therefore have a simple decision problem: a worker with outside option r accepts service requests from the firm iff $r \leq w$. A worker who decides to accept requests from the firm is called a *viable* worker. The total mass of viable workers is therefore equal to $mF(w)$, which we typically denote by λ, when m, w, F are clearly known from the context. We sometimes refer to a viable worker as a worker who has *joined* the firm.

The set of viable workers is partitioned into two subsets, namely *busy* and *idle* workers. The busy workers are those who are currently serving customers. The idle workers are the rest of the viable workers. We denote the mass of busy and idle workers with b, i, respectively.

Customers

Each customer has a valuation v for the service, which is drawn independently from a CDF G. A customer requests service (i.e., *joins* the firm) iff $v > p + c(i)$, where p is the price posted by the firm, and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a cost function where $c(i)$ denotes the cost incurred by the customer given that there are i idle workers available in the pool at the time of the customer's request. For example, one may consider $c(i)$ as the customer's waiting cost. We will assume that c is a decreasing convex function. The *payoff* of the customer from joining the firm is equal to $v - p - c(i)$.

For any fixed p, w offered by the firm, there is a unique rate of customers who join the firm, namely k. To determine k, we write the *market clearing condition* according to which the rate of service supplied equals the rate of service demanded:

$$k = 1 - G(p + c(mF(w) - k)). \tag{3.1}$$

On the right-hand side we have the rate of customers who join the firm, i.e., $\mathbb{P}_{v \sim G}\left[v > p + c(i)\right]$. Note that the left-hand side is strictly increasing in k, whereas the right-hand side is decreasing in k (holding all other variables fixed). This implies that there is a unique k satisfying the above equation. We typically write this k as a function of p, w and denote it by $k(p, w)$.

[3]Section 7.1 considers a different assumption under which the worker earns the outside option r only when she never works the firm. While similar qualitative findings hold under this alternative assumption, the original assumption makes the analysis simpler.

We use $k(p, w), \lambda(p, w)$ respectively to denote the rate of customers who join the firm and the mass of viable workers under price p and wage w. When p, w are clearly known from the context, we use the brief notation k, λ.

The firm

The firm posts price and wage so as to maximize its "objective function." The main objective functions that we consider for the firm are profit-maximization and throughput maximization.

Profit-maximizing firm. Under a fixed choice of p, w, the firm's profit function is defined by $\Pi(p, w) \equiv (p - w) \cdot k(p, w)$. The firm's objective is to maximize profit by choosing price and wage, i.e., the firm's problem is defined by

$$\begin{aligned} &\max_{p,w \geq 0} \Pi(p, w) \\ &\text{s.t. } k(p, w) \leq mF(w). \end{aligned} \tag{3.2}$$

(3.2) is a capacity constraint that states that the mass of viable workers has to be at least as much as the rate of customers who join. The optimal solution to the firm's problem is called the *profit-maximizing monopoly equilibrium* or briefly, when the firm's objective is clearly known from the context, the *monopoly equilibrium*. We say a profit-maximizing monopoly equilibrium *exists at* m when the firm serves a positive rate of customers at its optimal solution when given a labor pool of size m. An equilibrium is *nonbinding* if the capacity constraint does not bind at that equilibrium, i.e., when the number of idle workers is positive.

Throughput-maximizing firm. A *throughput-maximizing* firm chooses p, w to maximize $k(p, w)$ subject to the capacity constraint and the constraint $p \leq w$. The optimal solution to the firm's problem is called the *throughput-maximizing monopoly equilibrium* or briefly, when the firm's objective is clearly known from the context, the *monopoly equilibrium*. We say a throughput-maximizing monopoly equilibrium exists at m if there exists p, w with $p \leq w$ for which $k(p, w) > 0$. We note that a throughput-maximizing firm always chooses $p = w$.

When the firm's objective is clearly known from the context, we use $p(m), w(m)$ to denote the price and wage and $k(m)$ to denote the rate of customers who join the firm at a monopoly equilibrium.

Assumptions

Suppose that $F, G : [0,1] \to [0,1]$ are full-support strictly increasing CDFs and are of the class $\mathbf{C^4}$.[4] Furthermore, throughout the paper, we assume that F has a decreasing PDF, i.e., there are fewer workers with higher outside options.

We assume that the function $c : [0, \infty) \to [0, \infty]$ is of the class $\mathbf{C^4}$, decreasing, strictly convex, and that $c(0) = 1$. We call such a function a *regular* cost function.

Observe that under the assumption $c(0) = 1$, no customers will join the firm if there are no idle workers available, because the support of G is the unit interval. Therefore, this assumption ensures that all monopoly equilibria are nonbinding. Convexity of c has a simple interpretation: each additional idle worker decreases the waiting cost less than the previous one does.[5]

We discuss the assumptions further in Section 7, together with their relaxations and different variations of the model.

Existence and uniqueness of the equilibrium. There exists a *market failure threshold*, typically denoted by $\underline{m}$, such that profit-maximizing and throughput-maximizing monopoly equilibria exist and are generically unique[6] when $m > \underline{m}$ (e.g., see Theorem 4.1). Whenever we refer to such equilibria in a formal statement, the proof for their existence and uniqueness is included.

We typically use $p(m), w(m), k(m)$ to denote the equilibrium levels of price and wage, and the rate of customers who join the firm as a function of m. When m is clearly known from the context, we sometimes use the notations p^*, w^*, k^*, respectively.

4 Expanding the labor pool

Expanding the labor pool is one means through which platforms can improve service quality for customers. In a classical market place without externalities, the workers' average welfare and equilibrium wage is generically decreasing in the size of the labor pool (e.g., this is

[4] We recall that a function is of the the class $\mathbf{C^n}$ if its first n derivative exist and are continuous.

[5] Convexity of the cost function means that having one more idle worker is much more effective in decreasing the waiting cost when there are 10 idle workers available, than when there are 1000 idle workers available.

[6] Conditional on existence, the existence of equilibrium *generically* holds, in the following sense. The firm's objective function is bounded and continuous. This guarantees the existence of at least one local maximum. Loosely speaking, there are generically no two local maxima that are also global maxima, which means that the global maximum is generically unique.

the prediction of the law of demand in labor markets). We show that this can change in on-demand service platforms with a *thin* labor pool, i.e., when m is not sufficiently large. Then, increasing the size of the labor pool can *increase* the workers' wage, their average employment time, and their average welfare (Theorem 4.1). We will also show that this effect hinges on the convexity of the cost function.

We begin this section with two examples for the effect of increasing the size of the labor pool on the equilibrium wage. In these examples we discuss the intuition behind the observed non-monotonicity in wage and the crucial role of the convexity of the cost function in this non-monotonicity. Then, we present the main theorem of this section.

4.1 Examples

In our first example, we plot the wage in a profit-maximizing monopoly equilibrium when the cost function is an exponential cost function, i.e., $c(i) = e^{-i}$. We observe that the equilibrium wage increases with the size of the labor pool when the market is thin (i.e., below a certain threshold for m). This, however, is not the case when there are no externalities (demonstrated in Figure 1) or when the cost function is concave (demonstrated in Section 7.4).

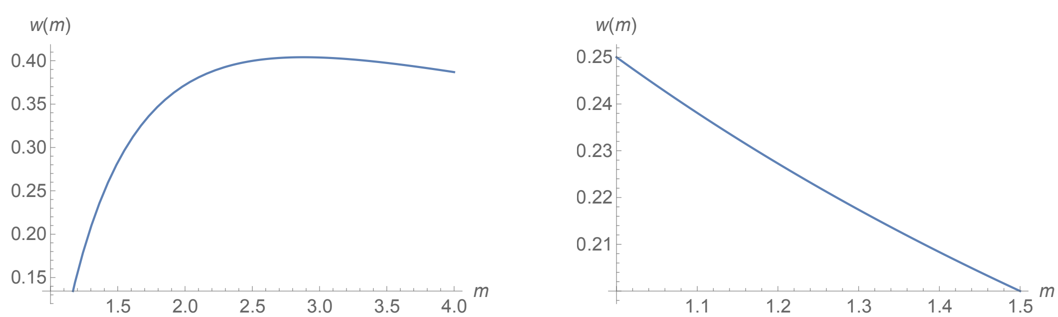


Figure 1: Equilibrium wage as a function of the size of the labor pool. F, G are the uniform distribution. In the left panel $c(i) = e^{-i}$ and in the right panel $c(i) = 0$ for all $i \geq 0$.

What is the intuition? As the size of the labor pool increases, the equilibrium wage is pressured by two forces. The first force works in favor of increasing wage: when m goes up, more workers join the firm for the same increase in wage.[7] This reduces the firm's marginal expenditure to acquire labor, which in turn pushes down the firm's marginal expenditure to improve service quality. (In our model, a higher service quality corresponds with a lower

[7]In a simple example, suppose that in a pool of 1000 workers, increasing the wage by 1 cent convinces, e.g., 10 more workers to join the firm. In a pool of 2000 workers, the same increase of 1 cent would convince, e.g., 20 more workers to join.

cost incurred by customers through the cost function c.) This creates a force in favor of increasing the wage, which we call the *upward force*.

The second force is the *downward force*: as m goes up, the number of idle workers goes up as well (holding all else fixed).[8] Therefore, an additional worker decreases the waiting cost less than when size of the labor pool is smaller (because of the convexity of the cost function). This force pushes up the firm's marginal expenditure to improve service quality, and thereby creates a force in favor of decreasing the wage. Whether the wage goes up or down depends on which force is stronger. When the labor pool is sufficiently large (relative to the arrival rate of customers), the service quality is high and the waiting cost is low. Additional idle workers do not decrease the waiting cost much compared to when the labor pool is small, because of the convexity of the cost function. Hence, the second force dominates the first force, and the equilibrium wage decreases with the size of the labor pool.

The same intuition holds when the firm's objective is throughput maximization, as we will see next. In the next example, we repeat the same exercise that derives the law of demand, but for the case of on-demand service platforms. Figure 2 presents the typical proof-by-picture for the law of demand. The equilibrium level of the wage is determined at the intersection of the inverse demand and supply functions, and it falls down as the labor supply goes up.

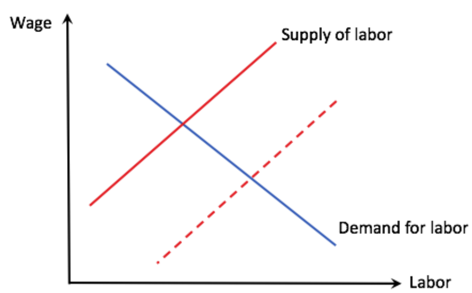


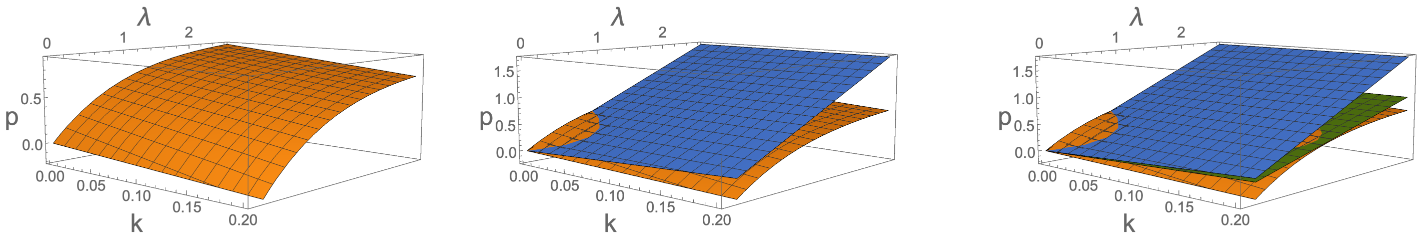
Figure 2: The law of demand

In Figure 3, we do the same exercise, but tailored for on-demand service platforms. Here, the inverse demand and inverse supply functions take two arguments as their input: the mass of viable drivers (λ) and the rate of customers who join (k).

Concavity of the inverse demand function is a direct consequence of the convexity of the cost function c (Figure 3(a)). The intersection of the inverse demand and inverse supply functions corresponds to a continuum of equilibria (Figure 3(b)). The throughput-maximizing

[8]This effect is intuitive when k is held fixed. Even when k changes endogenously as m changes (while holding price and wage fixed), the number of idle workers *always* increases for a marginal increase in m.

equilibrium corresponds to the equilibrium that maximizes k (observable in Figure 3(b)). When the size of the labor pool increases, the inverse supply function shifts down; then, the wage at the throughput-maximizing equilibrium can go up (as in Figure 3(c)). The intuition is similar to the case of the profit-maximizing firm: the same upward and downward forces pressure the equilibrium wage (which is also equal to the equilibrium price).



(a) Inverse demand function

(b) The inverse demand and inverse supply functions. The peak of the arc gives the throughput-maximizing equilibrium.

(c) The added surface (the middle surface in green color) corresponds to the inverse supply function at the higher m.

Figure 3

4.2 The welfare effect on workers

We will show that the equilibrium wage, workers' average welfare, and their average employment time increase with the labor pool when the labor pool is not too large. First, we formally define these terms.

When the equilibrium notion is clear from the context (i.e., throughput- or profit-maximizing equilibrium), we use $x(m)$ to denote the equilibrium value of a parameter x as a function of m, e.g., $w(m)$ denotes the equilibrium wage. Let the average employment time of workers be $e(m) \equiv \frac{k(m)}{\lambda(m)}$. Also, define the workers' average welfare as their per-round average earnings from wage and outside options:[9]

$$u^W(m) \equiv \frac{1}{F(w(m))} \cdot \int_0^{w(m)} (w(m) \cdot e(m) + r \cdot (1 - e(m)) \cdot F'(r) \, \mathrm{d}\, r. \tag{4.1}$$

For example, when F is the uniform distribution, the above expression simplifies to

$$u^W(m) = \frac{1}{2} \cdot \left(w(m) + \frac{k(m)}{m} \right), \tag{4.2}$$

[9] The theorem that we will present also holds for some other welfare-related notions, such as per round average earnings from wage.

which has a simple interpretation: the average worker always earns her outside option $r = \frac{w(m)}{2}$. In a fraction $e(m)$ of the time when she is serving a customer, she also earns an additional amount of $w(m) - r$.

Now we are ready to present the main theorem for this section.

Theorem 4.1. *Suppose that the firm's objective is throughput or profit maximization. Then, there exists* $\underline{m}$ *such that a monopoly equilibrium exists at* m *iff* $m > \underline{m}$*, and there exists* $\hat{m} > \underline{m}$ *such that for all* $m \in (\underline{m}, \hat{m})$*,* $w'(m), e'(m)$*, and* $(u^W)'(m)$ *are positive.*

In other words, when the market is thin, workers do not compete with each other, and increasing the labor pool increases their average welfare, wage, and average employment time. Intuitively, workers are *complements* in thin markets, whereas they can become *substitutes* otherwise.

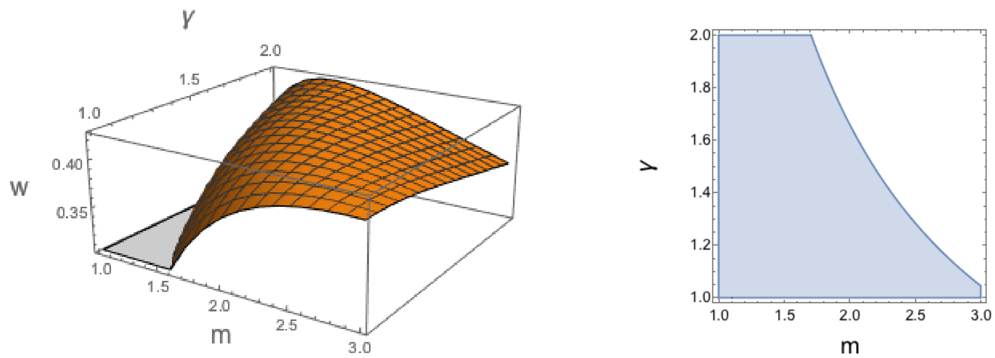
Section 7 performs robustness checks for this theorem under different assumptions, e.g., in a model where even idle workers lose their outside option when they join the firm. There, we see that the complementarity effect among workers is in fact magnified compared to the main setup, in the sense that the wage and average welfare remain increasing in m over a larger interval. The details are discussed in Section 7.

5 Improving matching technology

Aside from increasing the labor pool, platforms could improve service quality also by improving their resource allocation strategies. As in Section 4, we ask how would this affect the workers. More precisely, we study the effect of improved *matching technology*, i.e., improving the firm's matching algorithm so that service quality goes up (i.e., the cost incurred by customers goes down, e.g., due to lower waiting times), given the same labor supply. First, we present a simple example, where we observe that improving the matching technology increases the wage and workers' average welfare in thin markets and decreases these parameters otherwise. In other words, matching technology complements labor in thin markets, and substitutes it otherwise. We then discuss the intuition for this effect, set up a general model for improving matching technology, and present the main theorem of this section.

5.1 Example

Let F, G be the uniform distribution over the unit interval. Also, let $c(x) = e^{-\gamma x}$. In this example, we study the effect of increasing γ (i.e., improving the matching technology) on the wage in profit-maximizing monopoly equilibria. In Figure 4(a), we plot the equilibrium wage while varying m and γ. The shaded area in Figure 4(b) is where the derivative of equilibrium wage with respect to γ is positive. We observe that for any fixed γ, there exists a threshold $\hat{m}_\gamma$ such that the equilibrium wage increases with γ iff $m < \hat{m}_\gamma$. Also, we observe that the threshold $\hat{m}_\gamma$ decreases in γ.[10]



(a) Equilibrium wage.

(b) Wage increases with γ in the shaded area.

Figure 4

Before extending the above example to more general cost functions and distributions, we discuss the intuition. Consider the following thought experiment: Fix a monopoly equilibrium, and let p^*, w^*, k^*, i^* denote the equilibrium parameters, as defined previously. Hold price and wage fixed, and then improve the matching technology. This would correspond to changing the cost function c to a new cost function d such that $d(i) < c(i)$ holds for all positive i. Under the new matching technology, the number of customers who join the firm and the number of idle workers would change. Let $\tilde{i}$ and $\tilde{k}$ respectively denote the new parameters.

The core of this thought experiment is based on the following fact: under the new matching technology, the equilibrium wage increases iff $c'(i^*) > d'(\tilde{i})$. (Note that both $c'(i)$ and $d'(i)$ are negative numbers, each measuring how much the waiting cost would decrease for an additional idle worker.) In other words, the equilibrium wage increases iff adding one more idle worker decreases the waiting time more under the new technology than under the old technology. To understand whether this would hold, we should look at the two effects

[10] For a rough intuition for the latter effect, one can interpret improving the matching technology and expanding the labor pool both as means of improving the service quality: the higher the service quality, the less improving it benefits workers.

involved in the comparison $c'(i^*) > d'(\tilde{i})$: first, the argument of the functions change and, second, the functions themselves change. These effects pressure the equilibrium wage in two ways, as discussed below.

First effect: the arguments change. When technology is improved, service quality goes up, and therefore customer demand for the service goes up. Hence, the number of idle workers goes down, i.e., $\tilde{i} < i^*$. This creates a force that pushes down the firm's marginal expenditure to improve service quality (because, all else being equal, improving service quality is cheaper when the number of idle workers is smaller). This force, hence, pressures the equilibrium wage upward.

Second effect: the functions change. If $c'(i^*) < d'(i^*)$, then an additional idle worker decreases the waiting cost less under the new matching technology than under the old matching technology. When this inequality holds, it creates a force that pushes up the firm's marginal expenditure to improve service quality under the new technology, which pressures the equilibrium wage downward.

For example, when the functions c', d' satisfy the single crossing property, then $c'(i) < d'(i)$ holds for sufficiently large i. Hence, for sufficiently large m, i^* would be sufficiently large, $c'(i^*) < d'(i^*)$ holds, and the second effect exists. Otherwise, when m is small, $c'(i^*) < d'(i^*)$ does not hold because of the single-crossing property, and the second effect does not exist.

So far we have discussed the intuition behind the opposing forces that pressure the equilibrium wage when matching technology is improved. We next present the main theorem of this section, and then provide intuition on why the upward force dominates the downward force in thin markets.

5.2 Improved matching technology

We set up a general model of matching technology by allowing the cost function to depend on the technology level, which we denote by $\gamma \in [0, \infty)$. Let the function $c : \mathbb{R}_+^2 \to \mathbb{R}_+$ denote the cost function, with $c(\gamma, i)$ being the customer's waiting cost when there are i idle workers available at level γ of technology. A higher value of γ will correspond to a "better" matching technology.

We assume that (i) c is *smooth*: it is continuous and its partials with respect to its arguments exist and are continuous, (ii) the function $c(\gamma, \cdot)$ is a regular cost function for all γ, and (iii) cost goes down with technology: for any $\gamma_1 < \gamma_2$ and any positive i, $c(\gamma_1, i) >$

$c(\gamma_2, i)$. Assumptions (i) and (ii) have been discussed before. In particular, the regularity of the cost function ensures that no customers would join the firm if no idle workers are available. Assumption (iii) ensures that at a higher level of technology, the cost incurred by the customers is lower (e.g., because of lower waiting times).

Our next theorem shows that, under these assumptions, the workers' average welfare, the wage, and their average employment time increase with technology when the labor pool is not sufficiently large. To state the theorem, we let the variables $p(m, \gamma), w(m, \gamma), k(m, \gamma)$ denote the equilibrium values of these parameters as functions of m, γ. Also, define $e(m, \gamma) \equiv \frac{k(m,\gamma)}{\lambda(m,\gamma)}$. For a function $x(m, \gamma)$, we use the notation $x_i(m, \gamma)$ to denote the partial of $x(m, \gamma)$ with respect to its i-th argument, for $i \in \{1, 2\}$.

Theorem 5.1. *Suppose that the firm's objective is throughput or profit maximization. Then, for any $\gamma > 0$, there exists $\underline{m}_\gamma$ such that a monopoly equilibrium exists iff $m > \underline{m}_\gamma$. Furthermore, when $c_{1,2}(\gamma, \underline{m}_\gamma) \neq 0$, there exists $\hat{m}_\gamma > \underline{m}_\gamma$ such that for all $m \in (\underline{m}_\gamma, \hat{m}_\gamma)$, $w_2(m, \gamma), e_2(m, \gamma)$, and $(u^W)_2(m, \gamma)$ are positive.*

Intuitively, the theorem shows that matching technology complements labor in thin markets, but can substitute it otherwise.

Next, we discuss the role of the condition $c_{1,2}(\gamma, \underline{m}_\gamma) \neq 0$ and its connection with the single-crossing condition discussed in Section 5.1. Because of the regularity of the cost function, this condition implies that $c_{1,2}(\gamma, \underline{m}_\gamma) < 0$. This means that the second effect on the equilibrium wage (discussed in Section 5.1) does not pressure the equilibrium wage downward, when m is small. To see the intuition, recall from Section 5.1 that, when c', d' satisfy the single-crossing property, the second effect creates a downward pressure on wage when m is sufficiently large, but not when m is sufficiently small, as $c'(i(m)) > d'(i(m))$ holds for small m. The condition $c_{1,2}(\gamma, \underline{m}_\gamma) < 0$ essentially ensures the same but for a marginal change in the matching technology. This is less restrictive than single-crossing, in the sense that it does not rule out the possibility of the functions crossing more than once.

Finally, we remark that while the condition $c_{1,2}(\gamma, \underline{m}_\gamma) \neq 0$ is not necessary for the theorem to hold, it makes the proof significantly simpler.

6 The effect of competition

We study the effect of competition by comparing monopoly and duopoly equilibria. We start by setting up the duopoly model in Section 6.1, and then we present the results in

Section 6.2. One might expect that competition between firms should benefit the market participants. We find that this is not the case in thin markets. Then, although competition may remain beneficial to workers, it has an adverse effect on customers: the price would be higher and the customers' average welfare would be lower in the duopoly equilibrium compared to the monopoly equilibrium.

A simple intuition explains this adverse effect of competition. There are two main forces affecting the duopoly price. Competition over customers pressures the customer price downward, whereas competition over workers raises firms' costs, pressuring the customer price upward. The net effect of competition on price depends on the strength of these forces. When the labor pool is small, competition for workers dominates competition for customers, and the price is higher and the customers' average welfare is lower in the duopoly equilibrium than in the monopoly equilibrium.

6.1 Setup

The setup is similar to the monopoly setup, but with two firms. $\mathcal{F} = \{1, 2\}$ is the set of firms. When a firm f is clearly known from context, we sometimes use the notation $-f$ to refer to the other firm.

The high-level description of the game is as follows. Each firm chooses price and wage. The *payment profile of firm* f is the tuple $\mathbf{P}_f = (p_f, w_f)$. The *payment profile* $\mathbf{P}$ is defined by the tuple $(\mathbf{P}_1, \mathbf{P}_2)$. This payment profile then defines a subgame, which also determines the profits of the firms. In this subgame, agents (workers and customers) observe the payment profile and the decisions of the other agents, and make optimal decisions (about whether to join the firm) based on that information.[11] In a *steady-state subgame equilibrium under payment profile* $\mathbf{P}$, no agent benefits from changing her decision, taking the decisions of other agents as given. We will see that any payment profile $\mathbf{P}$ *induces* an *essentially unique* steady-state subgame equilibrium. Then, a duopoly equilibrium will be defined as the equilibrium of a game played by two firms whose actions are choosing price and wage.

We formally define the equilibrium notions next, and then the adverse effect of competition in Section 6.2.

[11] The decisions of all agents do not need to be common knowledge, so long as the values of some equilibrium parameters (such as the size of the pool and the rate of customers who join) are.

Agents

Workers. A worker can choose a subset of the firms from which to accept offers (service requests). This gives her $2 \times 2 = 4$ possible actions, one of which she chooses. We say a worker takes action S if she chooses to accept offers from the subset $S \subseteq \mathcal{F}$ of the firms. We say such a worker is *of type* S. We say a worker has *joined* firm f if she is of type S and $f \in S$.

Customers. The difference between this duopoly model and the monopoly model is that, here, customers' valuations are modeled by a joint distribution over the two firms. We thereby suppose that a customer's valuations are represented by (v_1, v_2) where v_f is the customer's valuation for firm f. We suppose that (v_1, v_2) are independently and identically drawn (iid) from a joint CDF $G : [0, 1]^2 \to [0, 1]$.

Immediately upon her arrival, each customer takes one of the following actions: requesting service from firm 1 (i.e., *joining* firm 1), joining firm 2, or not joining any firm. If the customer joins a firm, she will be served for a unit of time, after which she departs the market. If the customer decides not to join any firm, she immediately departs the market.

Compositions

Worker composition. A *worker composition* is a tuple (I, B) with

$$\begin{aligned} I &: 2^{\mathcal{F}} \to \mathbb{R}_+, \\ B &: 2^{\mathcal{F}} \times \mathcal{F} \to \mathbb{R}_+, \end{aligned}$$

where $I(S) \in \mathbb{R}_+$ denotes the mass of idle workers of type S and $B(S, f)$ denotes the mass of workers of type S busy at firm f. For consistency, we will use $i(S), b(S, f)$ to denote $I(S), B(S, f)$, respectively. We use $b(f)$ to denote the mass of all workers busy at firm f, i.e., $\sum_{S \ni f} b(S, f)$. We use $i(f)$ to denote the mass of all idle workers who accept offers from firm f, i.e., $\sum_{S \ni f} i(S)$.

Customer composition. A *customer composition* is a tuple $\mathbf{k} = (k_1, k_2)$ where k_f denotes the (steady-state) rate of customers who join firm f.

Composition. A composition $\mathbf{A}$ is a tuple $(\mathbf{k}, (I, B))$ where $\mathbf{k}$ is a customer composition and (I, B) is a worker composition.

Arrangement. An arrangement is a tuple $\Sigma = (\mathbf{P}, \mathbf{A})$ where $\mathbf{P}$ is a payment profile and $\mathbf{A}$ is a composition.

Payoffs

Customer's payoff. Under the arrangement $(\mathbf{P}, \mathbf{A})$, payoff of a customer from joining firm f is $v_f - p_f - c(i(f))$, where v_f is the valuation of the customer for firm f.

Worker's payoff Similar to the monopoly model, each worker has an outside option r, which is distributed from a distribution with CDF F. Given an arrangement $(\mathbf{P}, \mathbf{A})$, for any firm f, define $\gamma_f = \frac{k_f}{i(f)}$. This is just the steady-state rate at which an idle worker who accepts offers from firm f receives offers from f. For any action S, define $\gamma(S) = \sum_{f \in S} \gamma_f$. The interpretation for $\gamma(S)$ is that it is the steady-state rate at which a worker of type S receives offers. The payoff of a worker with outside option r who takes action S under arrangement $(\mathbf{P}, \mathbf{A})$ is

$$r \cdot \left(\frac{1}{1 + \gamma(S)} \right) + \sum_{f \in S} w_f \cdot \frac{\gamma_f}{1 + \gamma(S)}.$$

The above expression is just the steady-state earnings of a worker of type S per unit of time from wage and outside option. This quantity is derived from a straight forward calculation that computes the average time that a worker of type S remains idle or works at each of the firms (Lemma F.9).

Subgame equilibrium

An arrangement $\Sigma = (\mathbf{P}, \mathbf{A})$ is called a *subgame equilibrium* if the following conditions hold:

(i) **Customers optimize.** Each customer chooses the action that maximizes her payoff. In case the maximum payoff is attained by multiple actions, the customer chooses one of those actions uniformly at random.

(ii) **Workers optimize.** Each worker chooses the action that maximizes her payoff. If the maximum payoff is attained by multiple actions, the worker chooses the action with the smallest size. If there are multiple such actions, then the worker chooses one arbitrarily. (Nevertheless, we remark that there cannot be multiple such actions, as shown by Fact F.10 in the appendix)

(iii) **Balance equations.** Actions taken by the customers and workers *induce* the composition $\mathbf{A}$ in the steady state: assuming that at all times the composition is $\mathbf{A}$, then, at any time (i) for any $S \subseteq \mathcal{F}$, the rate of workers of type S who become idle equals the rate of workers of type S who become busy, (ii) for any $S \in \mathcal{F}$ and $f \in \mathcal{F}$, the rate of workers of type S who become busy at firm f equals the rate of workers of type S who finish serving a customer at firm f, and (iii) for any $f \in \mathcal{F}$, the rate of customers joining f equals the rate of customers leaving f.

When $\Sigma = (\mathbf{P}, \mathbf{A})$ is a subgame equilibrium, we sometimes say that Σ is a *subgame equilibrium under* $\mathbf{P}$, or a *subgame equilibrium induced by* $\mathbf{P}$.

Definition 6.1. *A nontrivial subgame equilibrium is a subgame equilibrium in which both firms serve a positive rate of customers. A subgame equilibrium that is not nontrivial is called trivial.*

Our main focus is on the nontrivial subgame equilibria, which, when they exist, are uniquely determined by the payment profile.

Proposition 6.2. *Any payment profile* $\mathbf{P}$ *induces at most one nontrivial subgame equilibrium.*

Any payment profile $\mathbf{P}$ induces at least one trivial subgame equilibrium: the subgame equilibrium in which no firm serves any customers. We call this the $\varnothing$ subgame equilibrium. There are at most two other trivial subgame equilibria: for each firm, there is at most one subgame equilibrium in which only that firm serves a positive rate of customers. (Lemma F.11). As we will see shortly, we focus on the nontrivial subgame equilibria for defining the notion of duopoly equilibrium.

Definition 6.3. *The steady-state profit of a firm* f *in a subgame equilibrium* Σ *is*

$$\Pi_f(\Sigma) \equiv k_f \cdot (p_f - w_f),$$

where k_f, p_f, w_f *respectively denote the steady-state rate of customers who join* f*, and the price and wage at firm* f *in* Σ.

We are almost ready to define the main equilibrium notion, the equilibrium of the game played by the two firms, which we call the *duopoly equilibrium.* In other words, a duopoly equilibrium is a payment profile $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ such that no firm f can increase its profit by deviating from $\mathbf{P}_f$ to another payment profile $\overline{\mathbf{P}}_f$. There is, however, one subtlety. Firm f may choose $\overline{\mathbf{P}}_f$ so that the payment profile $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ induces no nontrivial subgame equilibrium. In that case, which trivial equilibrium induced by $\overline{\mathbf{P}}$ should be selected? We consider two ways to address this: (i) defining a *selection rule* that determines the outcome of a deviation by selecting one of the trivial subgame equilibria when a nontrivial one does not exist under that deviation, and (ii) changing the game played by the firms by redefining a firm's action as choosing the quantity of customers that it would like to serve, rather than setting the price and wage (a well-known approach in the two-sided markets literature). In what follows, we take the former approach. Appendix I demonstrates that the second approach also results in the adverse effect of competition, as introduced earlier.

The selection rule

Although the selection rule will be used only to determine the outcome of a firm's deviation from the duopoly equilibrium, we define it more generally here. Given a payment profile $\mathbf{P}$, the selection rule selects a unique subgame equilibrium $\Sigma_{\mathbf{P}}$ that is induced by $\mathbf{P}$. We define $\Sigma_{\mathbf{P}}$ to be the unique steady-state subgame equilibrium that serves the highest rate of customers. The uniqueness is proved in the appendix by Lemma F.12: The lemma shows that $\Sigma_{\mathbf{P}}$ is the unique nontrivial subgame equilibrium under $\mathbf{P}$, if a nontrivial subgame equilibrium exists. Otherwise, $\Sigma_{\mathbf{P}}$ is the unique trivial subgame equilibrium under $\mathbf{P}$ in which the rate of customers served is the highest.[12]

Remark 6.4. *One of the simplest selection rules is the $\varnothing$ selection rule: the rule that selects the $\varnothing$ subgame equilibrium when a nontrivial subgame equilibrium does not exist. As the profits of both firms are 0 at the $\varnothing$ subgame equilibrium, this selection rule effectively eliminates deviations under which no nontrivial subgame equilibrium is induced by $\overline{\mathbf{P}}$. The selection rule that we have chosen does not eliminate such deviations, and therefore it leads to a stronger equilibrium notion.*

Duopoly equilibrium

A subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is a *duopoly equilibrium* if for any firm f and any payment profile $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$, $\Pi_f(\Sigma_{\mathbf{P}}) \geq \Pi_f(\Sigma_{\overline{\mathbf{P}}})$. A duopoly equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is called *symmetric* iff $\mathbf{P}_1 = \mathbf{P}_2$.

[12]Notably, this coincides with the subgame equilibrium that maximizes customers' welfare, where welfare is defined in the usual way as the integral over customers' payoffs.

We remark that both firms serve the same rate of customers at any symmetric duopoly equilibrium, as formally proved by Fact F.13.

6.2 The adverse effect of competition

We first demonstrate the adverse effect of competition through an example. Let $c(x) = e^{-\gamma x}$ with $\gamma > 0$ and let F be the uniform distribution over the unit interval. Recall that G denotes the joint distribution for a customer's valuations over the two firms, i.e., $(v_1, v_2) \sim G$ represents a customer's valuations for firms $1, 2$. We define G (implicitly) as follows:

$$v_1 = \sigma x + (1 - \sigma) y,$$
$$v_2 = \sigma x + (1 - \sigma)(1 - y),$$

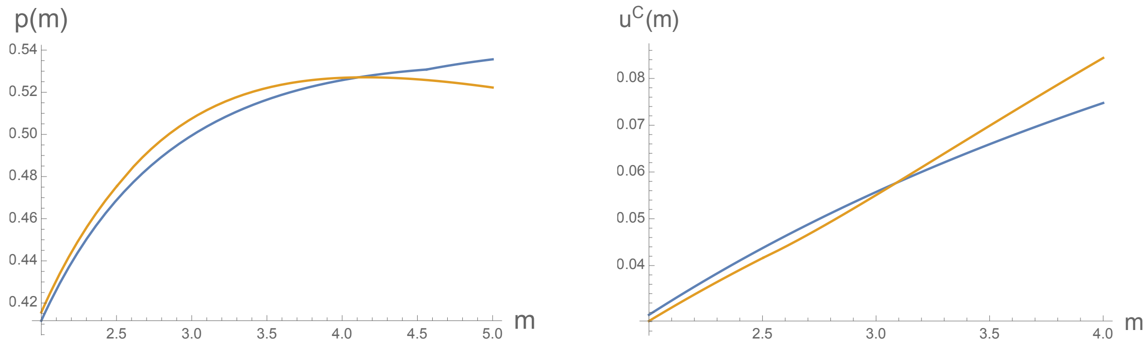
where x, y are iid uniform random variables with support over the unit interval and $\sigma \in (0, 1)$. The interpretation is that x is the *common value component* and y is the *idiosyncratic component*. σ determines the correlation between customers' preferences. The higher σ, the higher $\mathrm{corr}(v_1, v_2)$ would be. We will allow σ to be any constant greater than $1/2$, meaning that the weight of the common value component is larger than the weight of the idiosyncratic component.

We first assume that the firm's objective is profit-maximization, and compare the price and the customers' average welfare at the unique symmetric duopoly equilibrium to the price and the customers' average welfare at the unique monopoly equilibrium. (Whenever we refer to a monopoly or duopoly equilibrium in a formal statement, the proof for its existence and uniqueness will be included in the proof of that statement.) For brevity, from now on we refer to the symmetric duopoly equilibrium as the duopoly equilibrium.

Let $p_{\mathsf{duo}}(m)$ and $p_{\mathsf{mon}}(m)$ respectively denote the equilibrium price at the duopoly and monopoly equilibria. Similarly, let $u^C_{\mathsf{duo}}(m)$ and $u^C_{\mathsf{mon}}(m)$ respectively denote the customers' average welfare at the duopoly and monopoly equilibria. Customers' average welfare is defined in the usual way, as the integral of the customers' payoffs over the customers who join divided by the rate of customers who join.[13]

In this example, there exists $\hat{m}_1$ such that $p_{\mathsf{duo}}(m) > p_{\mathsf{mon}}(m)$ holds iff $m < \hat{m}_1$. Similarly, there exists $\hat{m}_2$ such that $u^C_{\mathsf{duo}}(m) < u^C_{\mathsf{mon}}(m)$ holds iff $m < \hat{m}_2$. To demonstrate, we have plotted these quantities in Figure 5.

[13]In defining $u^C_{\mathsf{duo}}(m)$, one may consider customers who join either firm, or only customers who join a fixed firm. The two definitions are identical, by symmetry of the duopoly equilibrium.



(a) $p_{\textsf{mon}}(m)$ and $p_{\textsf{duo}}(m)$. The monopoly price is lower than the duopoly price below a certain threshold for m.

(b) $u^C_{\textsf{mon}}(m)$ and $u^C_{\textsf{duo}}(m)$. The customers' average welfare is higher in the monopoly equilibrium than in the duopoly equilibrium below a certain threshold for m.

Figure 5: Price and customers' average welfare for $\sigma = 3/4$ and $\gamma = 1$.

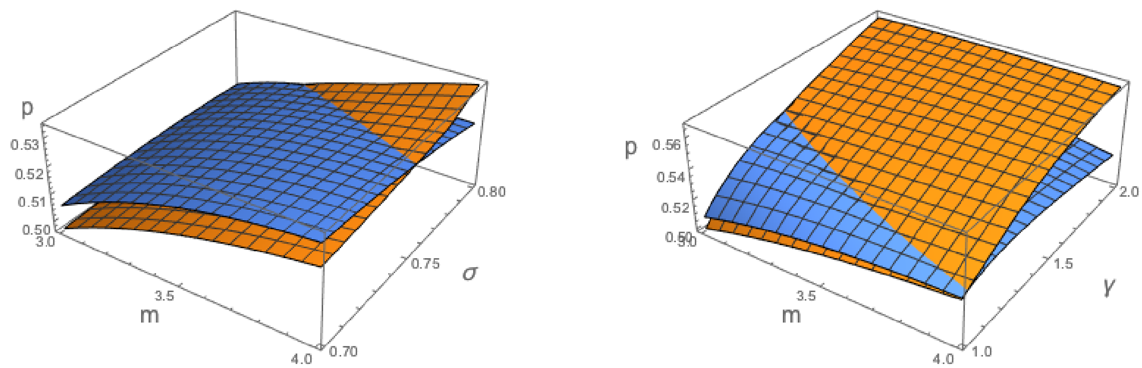
When the labor pool is not sufficiently large, the price is higher in the duopoly equilibrium and the customers' average welfare is lower. We dub this effect the *adverse effect of competition*. Before generalizing the observation in this example, we explain the intuition. The same intuition holds in a more general setting (Theorem 6.5).

The rough intuition is that two forces are pressuring the price. There is a *downward force* that pressures the price downward, and is derived by competition over customers. There is also an *upward force* that pushes up the wage and, thereby, the price. The upward force is derived by competition for workers. The duopoly price is greater than the monopoly price when the upward force is stronger than the downward force, i.e., when competition for workers *dominates* competition for customers. This happens in thin markets, where the labor pool is not sufficiently large.

Theorem 6.5. *Let c be a regular cost function. Then, there exists $\underline{m} > 0$ such that, when $m \le \underline{m}$, there exists no throughput- or profit-maximizing monopoly or duopoly equilibrium. Also, there exists $\hat{m} > \underline{m}$ such that for all $m \in (\underline{m}, \hat{m})$, (i) unique throughput and profit-maximizing monopoly and duopoly equilibria exist, and (ii) whether the firm's objective is profit or throughput maximization, it holds that $p_{\textsf{duo}}(m) > p_{\textsf{mon}}(m)$ and $u^C_{\textsf{duo}}(m) < u^C_{\textsf{mon}}(m)$, whereas $w_{\textsf{duo}}(m) > w_{\textsf{mon}}(m)$ and $u^W_{\textsf{duo}}(m) > u^W_{\textsf{mon}}(m)$. That is, competition has an adverse effect on customers, but not on workers.*

6.3 The effects of correlation and improving technology

We next study how the correlation between customers' preferences affects the adverse effect of competition. Figure 6(a) plots the monopoly and duopoly equilibrium prices while varying m and σ. (Recall that the higher σ, the higher the correlation between customers' preferences would be.) For any fixed σ, there is a value of m at which the duopoly price equals the monopoly price. Denote this value by $m(\sigma)$. For a fixed σ, the duopoly price is higher than the monopoly price iff $m < m(\sigma)$. Observe that $m(\sigma)$ is decreasing in σ (Figure 6(a)). The intuition is that as σ goes up, the competition for customers becomes stronger and, hence, it dominates the competition for workers at a lower value of m. The interpretation is that the adverse effect of competition on customers is alleviated when the correlation between customers' preferences increases.



(a) $m \in [3, 4]$, $\sigma \in [0.7, 0.8]$, and $c(i) = e^{-i}$

(b) $m \in [3, 4]$, $\sigma = \frac{3}{4}$, $\gamma \in [1, 2]$, and $c(i) = e^{-\gamma i}$

Figure 6: The orange surface (lighter color) and the blue surface (darker color) respectively give the monopoly and duopoly prices.

We detect a similar effect when the competition for workers is reduced: we demonstrate that the adverse effect of competition on customers is alleviated when the matching technology is improved (and hence, the competition over workers is reduced). Figure 6(b) plots the monopoly and duopoly equilibrium prices while varying m and γ, where higher levels of γ correspond to higher levels of matching technology. For any fixed γ, let $m(\gamma)$ denote the value of m at which the duopoly price equals the monopoly price. Observe that $m(\gamma)$ is decreasing in γ. The intuition is that as γ increases, the competition over workers becomes weaker (because the matching technology becomes stronger) and, consequently, the competition for customers dominates the competition for workers at a lower value of m.

7 Robustness checks and discussion

In this section we provide robustness checks for the main findings under alternative assumptions and models, and also discuss the role of the assumptions.

7.1 Workers lose their outside option if they join the firm

In our main setup a worker earns her outside option r whenever she is not busy serving a customer. Here we consider an alternative assumption.

Assumption 7.1 (The alternative assumption)**.** *Workers who join the firm completely lose their outside option, and hence do not earn it even when they are not busy serving a customer.*

Therefore, a worker would join the firm iff her outside option is smaller than the wage that she would earn per unit of time at the steady-state if she joins the firm. We will demonstrate that, under this assumption, the effect of increasing the labor pool on the workers' average welfare and wage remains similar to our main setup (Theorem 4.1), and that the complementarity effect between workers is in fact magnified.

The alternative assumption makes the model less tractable. In Section K.1 in the appendix we provide a counterpart for Theorem 4.1 that shows when the distributions F, G are the uniform distribution over the unit interval and the firm's objective is throughput maximization, the workers' average welfare and wage increases with size of the labor pool when m is small.

In Section K.2 in the appendix we consider a profit-maximizing firm. When F, G are the uniform distribution, we can simplify the system of equilibrium-characterizing equations to a single equation with a single unknown, which we can solve numerically. Figure 7 plots the equilibrium wage as a function of m under Assumption 7.1 and also under the original assumption in our main setup. The left plot corresponds to Assumption 7.1 and the right plot corresponds to the assumption in the main setup. Non-monotonicity in wage is present under both assumptions. We also observe that wage remains increasing in m over a larger interval in the left plot, suggesting that the complementarity effect between the workers is magnified under Assumption 7.1.

The intuition is simple: under that assumption, fewer workers join the firm at any fixed wage level (because workers lose their outside option if they join). Hence, loosely speaking, this assumption makes the market effectively thinner. Thereby, the effect corresponding to thin markets persists over a larger interval under this assumption.

We make similar observations about the workers' average welfare (Section K.2).

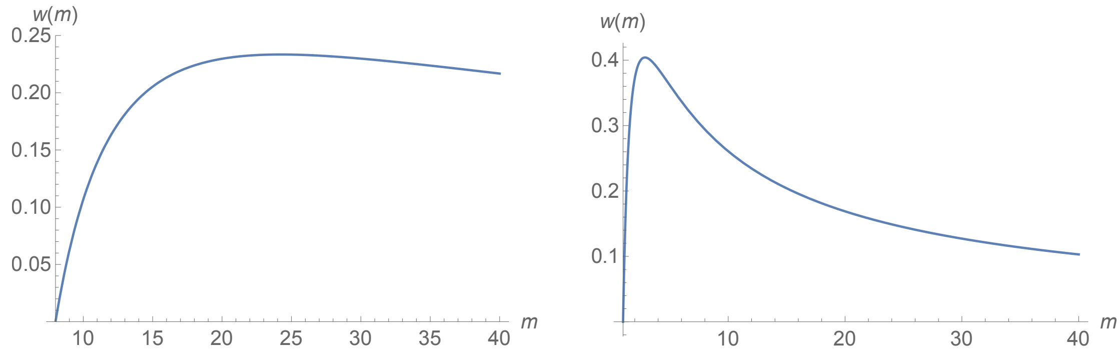


Figure 7: Equilibrium wages as functions of m for when F, G are the uniform distribution over $[0, 1]$ and $c(i) = e^{-i}$. The left and right plots respectively correspond to the alternative and original assumptions. In both figures the lowest value on the horizontal axis is the market failure threshold.

7.2 Service delays: the case of ride-sharing markets

Ride-sharing platforms are a natural example of on-demand service platforms. We consider a variation of our main setup by incorporating customer pickup times—which would fit this particular application—to evaluate the robustness of our main findings in the modified setup.

We suppose that a customer who requests service is served with a delay $t(i)$, where i denotes the number of idle workers available upon the arrival of the customer and $t : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function. This is the amount of time that the worker (i.e., the driver) is traveling to pick up the customer. The worker earns neither the wage nor her outside option in this time period. She only earns the (per unit of time) wage w after the customer is picked up. The waiting cost of the customer is given by a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ as follows: $h(t)$ determines the waiting cost of a customer when she is served with a delay t. Define $c(i) = h(t(i))$, which corresponds to the waiting cost of a customer when there are i idle workers available upon her arrival in the market. All else remains the same as in our main setup. In particular, we recall that c is assumed to be a regular cost function. The interpretation is that no customer will request service if there are no idle workers available.

We will provide a counterpart to Theorem 4.1 in this setting, showing that *workers are complements* in thin markets. The notion of average employment time is defined as in Theorem 4.1. The notion of workers' average welfare is similar to the one defined in that theorem, except that here we should also account for the pickup times; hence, we define

$$u^W(m) \equiv \frac{1}{F(w(m))} \int_0^{w(m)} (w(m)e(m) + r(1 - e(m)(1 - t(i(m))))) \cdot F'(r)\,\mathrm{d}\,r,$$

where we recall that the parameters $e(m), i(m)$ respectively denote the steady-state employment time per unit of time and the mass of idle workers at the monopoly equilibrium as a function of m.

Theorem 7.2. *There exists $\underline{m}$ such that a profit-maximizing monopoly equilibrium exists at m iff $m > \underline{m}$, and there exists $\hat{m} > \underline{m}$ such that for all $m \in (\underline{m}, \hat{m})$, $w'(m), e'(m)$, and $(u^W)'(m)$ are positive.*

7.3 The distribution of workers' outside options

In our main setup, we assume that the distribution of workers' outside options (F) has a decreasing PDF. While this assumption simplifies the proofs, it is not a necessary assumption for our main findings. For example, Figure 8 demonstrates the non-monotonicity of wage in the size of the labor pool for when this assumption does not hold.

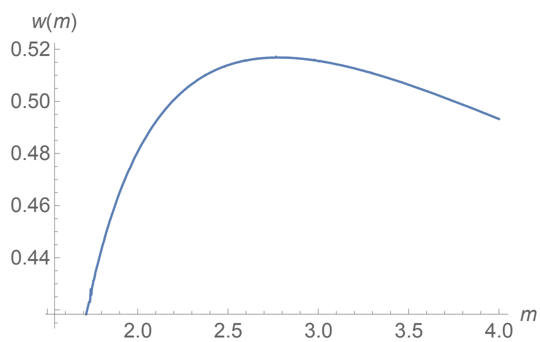


Figure 8: Equilibrium wage as a function of m for when the CDF of F is $f(x) = x^{3/2}$ for $x \in [0, 1]$, G is the uniform distribution over $[0, 1]$, and $c(i) = e^{-i}$.

7.4 Convexity of the cost function c

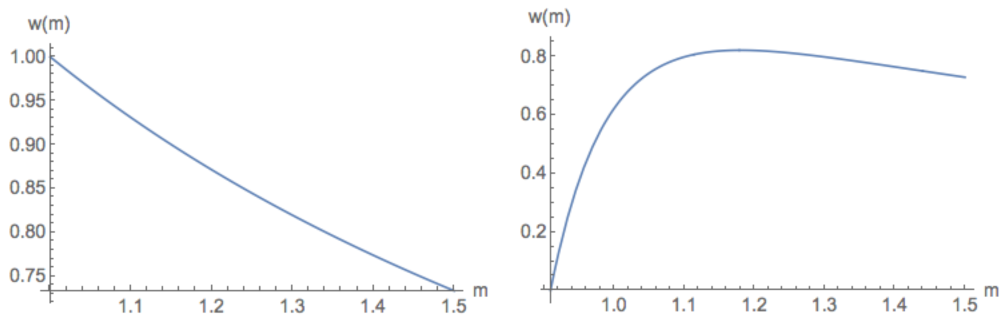
Convexity of the cost function is a main derivative of the complementarity effects when the labor pool is small. To clarify, we start with a monopoly example. Suppose that the cost function c belongs to the family

$$\mathcal{C} = \{c_\gamma : \gamma > 0\},$$

with $c_\gamma(i) \equiv (\max\{0, 1-i\})^\gamma$. Such functions are concave for $\gamma < 1$, affine for $\gamma = 1$, and convex for $\gamma > 1$. The main goal in this example is to demonstrate the role of convexity of the cost function by letting γ vary from below 1 to above 1.

In Figure 9, we compare equilibrium wages as a function of m for when c is concave or convex. For $\gamma < 1$, the prediction by the law of demand holds: equilibrium wage decreases with the size of the labor pool, m. For $\gamma > 1$, the equilibrium wage *increases* with m when m is below a certain threshold.

For the case of concave cost functions, the monopoly solution would not be an interior solution of the firm's profit-maximization problem, as demonstrated in Section L of the Appendix. For example, if F, G are the uniform distribution and c is concave, the monopolist employs just enough workers to provide the lowest possible waiting time for customers, which would imply that the equilibrium wage decreases with the size of the labor pool.



(a) Concave or affine c: any $\gamma \leq 1$.

(b) Convex c: $\gamma > 1$

Figure 9

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Appendices

A Monopoly: preliminaries and basic properties

The following results hold under the assumptions of our main setup, as stated in Section 3.

Lemma A.1. *The profit function* $\Pi(p, w) \equiv (p-w) \cdot k(p, w)$ *is continuous and bounded from above by* 1.

Proof. The proof for boundedness follows from the fact that $p-w \leq 1$ and $k(p, w) \leq 1$. The proof for continuity follows from the market-clearing condition:

$$k = 1 - G(p + c(mF(w) - k)).$$

Note that $k(p, w)$ is the root of the above equation. Also, observe that the right-hand side is decreasing in k, p and increasing in w. The right-hand side is also continuous in all of these variables. This implies that $k(p, w)$ is continuous in (p, w), which concludes the proof. □

Lemma A.2. *The partials* $\Pi_p(p, w)$ *and* $\Pi_w(p, w)$ *exist when* $k(p, w) > 0$.

Proof. First, we compute

$$\begin{aligned} \Pi_p(p, w) &= k(p, w) + (p - w)k_p(p, w) \\ \Pi_w(p, w) &= -k(p, w) + (p - w)k_w(p, w). \end{aligned}$$

Hence, to conclude the proof, it suffices to show that the partials $k_p(p, w)$ and $k_w(p, w)$ exist when $k(p, w) > 0$. We compute the closed-form expressions for these partials in file "partials":

$$\begin{aligned} k_p(p, w) &= \frac{G'(c(mF(w) - k) + p)}{c'(mF(w) - k)G'(c(mF(w) - k) + p) - 1} \\ k_w(p, w) &= \frac{mF'(w)c'(mF(w) - k)G'(c(mF(w) - k) + p)}{c'(mF(w) - k)G'(c(mF(w) - k) + p) - 1}. \end{aligned}$$

Observe that both denominators are negative. Hence, the above partials exist. This concludes the proof. □

Lemma A.1 and Lemma A.2 allow us to write the equilibrium-characterizing condition for the profit-maximizing monopoly equilibrium as follows.

Proposition A.3. *Any nonbinding profit-maximizing equilibrium must satisfy*

$$\begin{cases} k = 1 - G(p + c(mF(w) - k)), \\ k = -k_p(p, w) \cdot (p - w), \\ c'(i) = \frac{-1}{mF'(w)}, \end{cases} \tag{A.1}$$

where p, w, k respectively denote the equilibrium values of price, wage, and the rate of customers who join the firm.

Proof. The first equation is just the market-clearing condition stating that the rate of service supplied is equal to the rate of service demanded. The second equation is the firm's first-order condition with respect to price. The third equation is derived as follows. The firm's first-order condition with respect to wage is $k = k_w(p, w) \cdot (p - w)$. Equating the right-hand side of this equation with the right-hand side of the second equation gives $-k_p(p, w) = k_w(p, w)$. Simplifying this equation gives the third equation, as we show next. We compute the closed-form expressions for these partials in file "partials" by implicit differentiation from the first equation. Plugging the closed-form expressions in the latter equality gives

$$-\frac{G'(c(mF(w) - k) + p)}{c'(mF(w) - k)G'(c(mF(w) - k) + p) - 1} = \frac{mF'(w)c'(mF(w) - k)G'(c(mF(w) - k) + p)}{c'(mF(w) - k)G'(c(mF(w) - k) + p) - 1}.$$

Since the CDF G has full support, then it must hold that $mF'(w)c'(mF(w) - k) = 1$, which is the third equation.

$\square$

Definition A.4. *Consider a payment profile (p, w). Let k be the rate of customers requesting service under (p, w). Then, (p, w) is called a* standard *payment profile if it satisfies $c'(mF(w) - k) = \frac{-1}{mF'(w)}$.*

As shown above, any payment profile at the monopoly equilibrium must be a standard payment profile.

Proposition A.5. *Any nonbinding throughput-maximizing equilibrium must satisfy*

$$\begin{cases} k = 1 - G(p + c(mF(p) - k)), \\ c'(i) = \frac{-1}{mF'(p)}, \end{cases} \tag{A.2}$$

where p denotes the equilibrium price (which is equal to wage) and k denotes the rate of customers who join the firm.

Proof. The first equation is just the market-clearing condition stating that the rate of service supplied is equal to the rate of service demanded. The second equation is the (simplified) firm's first-order condition with respect to price. It is derived by implicit differentiation with respect to p from the first equation and setting the result to 0. In file "msm-prelim" we use implicit differentiation with respect to p from the first equation and compute

$$k'(p) = \frac{mF'(p)c'(mF(p)-k)G'(c(mF(p)-k)+p) + G'(c(mF(p)-k)+p)}{c'(mF(p)-k)G'(c(mF(p)-k)+p) - 1}.$$

The firm's first-order condition is $k'(p) = 0$. Since G has full support, then by the above equation we must have $mF'(p)c'(mF(p) - k) = 0$, which is the second equation in the statement of the proposition. □

It is helpful to write the equilibrium-characterizing equations for a profit-maximizing firm in a different form: in terms of allocation quantities rather than prices. To this end, suppose that $H \equiv F^{-1}$ and $J \equiv G^{-1}$. Also, recall that $\lambda = mF(w)$ denotes the mass of viable workers, and k denotes the rate of customers who join the firm. The firm's problem would then be choosing the quantities λ, k so that its profit is maximized, while the condition $k = 1 - G(p + c(\lambda - k))$ is satisfied. Observe that this equation allows us to write p in terms of k, λ as follows: $p = J(1-k) - c(\lambda - k)$. Also, observe that $w = H(\frac{\lambda}{m})$.

Lemma A.6. *Any profit-maximizing monopoly equilibrium satisfies the following conditions:*

$$k \cdot (c'(\lambda - k) - J'(1-k)) - c(\lambda - k) - H(\lambda/m) + J(1-k) = 0, \tag{A.3}$$

$$c'(\lambda - k) + \frac{H'(\lambda/m)}{m} = 0. \tag{A.4}$$

Proof. We can write the firm's profit function as

$$\Pi(\lambda, k) = k \cdot (J(1-k) - c(\lambda - k) - H(\lambda/m)).$$

This allows us to write the firm's FOC for the choice of k:

$$k \cdot (c'(\lambda - k) - J'(1-k)) - c(\lambda - k) - H(\lambda/m) + J(1-k) = 0,$$

which is obtained by setting the partial derivative of the profit function with respect to k to 0, i.e., setting $\Pi_k(\lambda, k) = 0$. The above equation is the same as (A.3).

We can rewrite the equation $c'(i) = \frac{-1}{mF'(w)}$ (given by (A.2)) as

$$c'(\lambda - k) + \frac{H'(\lambda/m)}{m} = 0. \tag{A.5}$$

This is the same equation as (A.4). The proof is complete. □

B Proof of Theorem 4.1: profit-maximizing firm

This section contains the proof for Theorem 4.1 for the case when the firm's objective is profit maximization.

Let $S(m)$ denote the system of equations given by (A.3) and (A.4), for a given m. Let the function $X(m, \lambda, k)$ and $Y(m, \lambda, k)$ denote the LHS of equations (A.3) and (A.4), respectively. Also, let $m_0 = \frac{1}{-c'(0)F'(0)}$. We will show that $\underline{m} = m_0$.

Lemma B.1. *There exists a monopoly equilibrium at m iff $m > m_0$.*

Proof. We say that a pair (p, w) is *feasible at m* if $k(p, w) > 0$ holds when the size of the labor pool is m. We say that m is *feasible* if there exists a pair (p, w) which is feasible at m. We will show that m is feasible iff $m > m_0$. This would prove the claim.

Claim B.2. *Suppose (p, p) is feasible for a fixed m. Then, there exists $w < p$ such that (p, w) is also feasible.*

Proof. Let $\hat{k} = k(p, p)$. There exists $w < p$ such that $p + c(mF(w) - \hat{k}) < 1$, by the continuity of c and F. Then, observe that there exists a unique value of k, namely $\tilde{k}$, that solves the equation $k = 1 - G(p + c(mF(w) - k))$, because the left-hand side is strictly increasing and the right-hand side is decreasing in k. Furthermore, $\tilde{k} \in [0, \hat{k}]$. Therefore, by the choice of w, we have $p + c(mF(w) - \tilde{k}) < 1$, which implies that $\tilde{k} > 0$. Therefore, (p, w) is feasible. □

Next, we show that if $m = \frac{1}{-c'(0)F'(0)} + \delta$ for some $\delta > 0$, then m is feasible. We prove this by proving the existence of a feasible (p, w). To do this, we show that there exists a p such that (p, p) is feasible; that is,

$$p + c(mF(p)) < 1. \tag{B.1}$$

Define the function $f(p) \equiv p + c(mF(p))$. Observe that $f(0) = 1$. We next show that $f'(0) < 0$, which would guarantee the existence of a positive p (sufficiently close to 0) that satisfies (B.1):

$$f'(p) = 1 + mF'(p)c'(mF(p))$$
$$\Rightarrow f'(0) = 1 + mF'(0)c'(0) = 1 + \left(\frac{1}{-c'(0)F'(0)} + \delta\right) F'(0)c'(0) = \delta F'(0)c'(0) < 0.$$

The last inequality is implied by the assumption $F''(r) \leq 0$, which implies $F'(0) > 0$. Given that $f'(0) = 1$, $f'(0) < 0$ guarantees that there exists p arbitrarily close to 0 such that $p + c(mF(p)) < 1$.

It remains to show that any $m \leq \frac{1}{-c'(0)F'(0)}$ is not a feasible mass of workers. The proof is by contradiction. Suppose m is a feasible mass of workers. Therefore, a monopoly equilibrium exists. Let (p^*, w^*, k^*) denote the equilibrium parameters, and let $i^* = mw^* - k^*$. The monopoly equilibrium must satisfy $c'(i^*) = \frac{-1}{mF'(w^*)}$. Then, observe that

$$c'(i^*) = \frac{-1}{mF'(w^*)} \leq \frac{-1}{\frac{1}{-c'(0)F'(0)} \cdot F'(0)} = c'(0).$$

But then, strict convexity of c would imply that $c'(i^*) = c'(0)$. Therefore, $i^* = 0$, which also implies that $k^* = 0$, which is a contradiction.

□

For the rest of the proof, we extend the domains of the functions c, F, G so that: (i) their domains contain an interval $(-\epsilon, 0)$ for some $\epsilon > 0$, (ii) c remains in $\mathbf{C^4}$ and strictly convex in the extended domain, (iii) the functions F, G remain in $\mathbf{C^4}$ in the extended domain, and (iv) F' remains decreasing in the extended domain. It is straightforward to verify that such an extension exists.

Claim B.3. *There exists an open interval $I = (m_1, m_2)$ containing m_0 such that $S(m)$ has a solution at any $m \in I$. Furthermore, there exist unique continuously differentiable functions $\lambda(m), k(m) : \mathbb{R}_+ \to \mathbb{R}$ such that $(\lambda(m), k(m))$ is a solution to $S(m)$, for any $m \in I$.*

Proof. The proof is based on the Implicit Function Theorem. As we mentioned earlier, we apply the theorem to the system given by (A.3) and (A.4). First, recall that (A.3) and (A.4) are given by

$$k \cdot (c'(\lambda - k) - J'(1-k)) - c(\lambda - k) - H(\lambda/m) + J(1-k) = 0,$$

and

$$c'(\lambda - k) + \frac{H'(\lambda/m)}{m} = 0.$$

Also, recall that the left-hand sides of (A.3) and (A.4) are respectively denoted by $X(m, \lambda, k)$ and $Y(m, \lambda, k)$. Next, we show that $(\lambda, k) = (0, 0)$ is a solution to $S(m_0)$, i.e., $X(m_0, 0, 0) = 0$, and $Y(m_0, 0, 0) = 0$.

$$0 \cdot (c'(0) - J'(1)) - c(0) - H(0) + J(1) = 0,$$

which holds because $c(0) = 1$, $H(0) = 0$, and $J(1) = 1$. Also,

$$c'(0) + \frac{H'(0)}{m_0} = 0,$$

which holds because $m_0 = \frac{-1}{F'(0)c'(0)}$ and $H'(0) = \frac{1}{F'(0)}$.

To apply the implicit function theorem, we need to prove that the Jacobian

$$J(m, k, \lambda) = \begin{pmatrix} \frac{\partial X(m,\lambda,k)}{\partial \lambda} & \frac{\partial X(m,\lambda,k)}{\partial k} \\ \frac{\partial Y(m,\lambda,k)}{\partial \lambda} & \frac{\partial Y(m,\lambda,k)}{\partial k} \end{pmatrix} \tag{B.2}$$

is invertible at point $(m, \lambda, k) = (m_0, 0, 0)$. This is done in file "pool-thm-Jacobian". There, we compute the closed-form expression for the determinant of the above matrix at point $(m_0, 0, 0)$ as

$$\frac{(c''(0)m_0^2 + H''(0))\left(\frac{2H'(0)}{m_0} + 2J'(1)\right)}{m_0^2}.$$

Observing that $c''(0) > 0$, $H'(0) > 0$, $H''(0) \geq 0$ and $J'(1) \geq 0$ implies that the right-hand side is positive, and hence the matrix is invertible at point $(m_0, 0, 0)$.

The Implicit Function Theorem therefore applies, and there exist an open interval $I \ni m_0$ and unique continuously differentiable functions $\lambda(m), k(m)$ that solve the system $S(m)$ for all $m \in I$. Furthermore, the theorem implies the existence of $\lambda'(m)$ and $k'(m)$ for all $m \in I$. □

Note that in the above claim, we allow the functions $\lambda(m), k(m)$ to have a possibly negative range.

Claim B.4. *The following relations hold:*

$$\lim_{m\to m_0} k'(m) = 0, \qquad \lim_{m\to m_0} k''(m) > 0$$
$$\lim_{m\to m_0} \lambda'(m) > 0.$$

Furthermore, the limits

$$\lim_{m\to m_0} k'''(m), \lim_{m\to m_0} \lambda''(m), \lim_{m\to m_0} \lambda'''(m)$$

exist and are finite.

Proof. We prove the claim in file "pool-thm-limits", where we use implicit differentiation with respect to m from the system $S(m)$ to compute the closed-form expressions for the derivatives. We then use these closed-form expressions to compute the limits. These derivations are explained step by step in file "pool-thm-limits". We go over these derivations below.

We start by computing a closed-form expression for $k'(m)$. There, we use implicit differentiation with respect to m from the system $S(m)$, to compute

$$k'(m) = \frac{-\left(k\lambda c''(\lambda-k)H''\left(\frac{\lambda}{m}\right) + \frac{H'\left(\frac{\lambda}{m}\right)\left(m^2(k+\lambda)c''(\lambda-k)+\lambda H''\left(\frac{\lambda}{m}\right)\right)}{m}\right)}{\left(m^3\left(\left(c''(\lambda-k)+\frac{H''\left(\frac{\lambda}{m}\right)}{m^2}\right)\left(k\left(J''(1-k)-c''(\lambda-k)\right)-\frac{2H'\left(\frac{\lambda}{m}\right)}{m}-2J'(1-k)\right)+kc''(\lambda-k)^2\right)\right)},$$

where on the right-hand side k, λ denote $k(m), \lambda(m)$, respectively. Now, recall that $\lambda(m), k(m)$ are continuously differentiable functions. Hence, we have $\lim_{m\to m_0} k(m) = k(m_0)$ and $\lim_{m\to m_0} \lambda(m) = \lambda(m_0)$. Recall from the proof of Claim B.3 that $k(m_0) = \lambda(m_0) = 0$. Using this fact and the inequalities

$$c''(0) > 0, H'(0) > 0, H''(0) \geq 0, J'(1) > 0, \tag{B.3}$$

from the closed-form expression for $k'(m)$ implies that $\lim_{m\to m_0} k'(m) = 0$.

We then use implicit differentiation with respect to m from the system $S(m)$ to compute a closed-form expression for $k''(m)$ in file "pool-thm-limits" (which would not fit on this page). Using this expression and the facts that $\lim_{m\to m_0} k(m) = k(m_0)$, and $\lim_{m\to m_0} \lambda(m) = \lambda(m_0)$, we compute

$$\lim_{m\to m_0} k''(m_0) = \frac{H'(0)^2}{2m\left(m^2c''(0)+H''(0)\right)\left(H'(0)+mJ'(1)\right)}.$$

The facts that

$$c'(0) < 0, c''(0) > 0, H'(0) > 0, H''(0) \geq 0, J'(1) > 0 \tag{B.4}$$

imply that the right-hand side is positive. This shows that $\lim_{m\to m_0} k''(m) > 0$.

Similarly, we compute a closed-form expression for $\lim_{m\to m_0} k'''(m)$. (We recall that the computations are done in file "pool-thm-limits".) This expression is a fraction $\frac{A}{B}$ where A is a real number (with a closed-form expression as given in file "pool-thm-limits") and

$$B = 2m^2\left(m^2c''(0)+H''(0)\right)^3\left(H'(0)+mJ'(1)\right)^2.$$

By (B.4), we have $B > 0$, which implies that the limit $\lim_{m\to m_0} k'''(m)$ exists.

It remains to compute the limits regarding $\lambda(m)$. In file "pool-thm-limits" we use implicit differentiation with respect to m from the system $S(m)$ to compute

$$\lim_{m\to m_0} \lambda'(m) = \frac{H'(0)}{m^2c''(0)+H''(0)}.$$

By (B.4), the right-hand side is positive.

We also compute the closed-form expression for $\lim_{m\to m_0} \lambda''(m)$ as $\frac{C}{D}$ where C is a real number and

$$D = 2m\left(m^2c''(0)+H''(0)\right)^3\left(H'(0)+mJ'(1)\right).$$

By (B.4), $D > 0$, which implies that the limit $\lim_{m\to m_0} k'''(m)$ exists.

Finally, we compute the closed-form expression for $\lim_{m\to m_0} \lambda'''(m)$ as $\frac{E}{F}$ where E is a real number and

$$F = 2m^2\left(m^2c''(0)+H''(0)\right)^5\left(H'(0)+mJ'(1)\right)^2.$$

By (B.4), we have that $F > 0$, which implies that the limit $\lim_{m\to m_0} \lambda'''(m)$ exists. This concludes the proof.

□

Claim B.5. *There exists $m_3 > m_0$ such that $\lambda'(m) \neq 0$ for any $m \in (m_0, m_3)$.*

Proof. This is a consequence of Claim B.4 and the continuity of $\lambda'(m)$. (The continuity holds due to Claim B.3.) □

To prove Theorem 4.1, we will show that there exists $\hat{m} > m_0$ such that $e'(m) > 0$, $w'(m) > 0$, and $(u^W)'(m)$ hold for all $m \in (m_0, \hat{m})$. This will be shown in Proposition B.6, Proposition B.9, and Proposition B.10.

Proposition B.6. *There exists a threshold $\hat{m}$ such that for all $m \in (m_0, \hat{m})$, $e'(m) > 0$.*

Proof. First, we prove the following claims.

Claim B.7. *There exists $m_4 > m_0$ such that $e'(m)$ exists at all $m \in (m_0, m_1)$.*

Proof. Observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Because $\lambda(m) > 0$ for all $m > m_0$, and because $k(m), k'(m)$ and $\lambda'(m)$ exist and are finite for m sufficiently close to m_0 (by Claim B.4), $e'(m)$ exists and is finite for m sufficiently close to m_0.

□

Claim B.8. $\lim_{m \to m_0} e(m) = 0$.

Proof. First, observe that

$$\lim_{m \to m_0} e(m) = \lim_{m \to m_0} \frac{k(m)}{\lambda(m)} = \lim_{m \to m_0} \frac{k'(m)}{\lambda'(m)}.$$

L'Hôpital's rule is applicable here by Claim B.5. By Claim B.4, we have $\lim_{m \to m_0} k'(m_0) = 0$ and $\lim_{m \to m_0} \lambda'(m_0) > 0$. This concludes the proof.

□

Continuity of $e(m)$ at m_0 is ensured by Claim B.8. The rest of the proof is as follows. We will show that $\lim_{m \to m_0} e'(m)$ exists and is positive. This would imply that $e'(m_0)$ must also exist, and in fact,

$$e'(m_0) = \lim_{m \to m_0} e'(m).$$

(This is a consequence of L'Hôpital's rule. See, for example, [Wikipedia 2017] for a proof.) Once we have shown this, the proof is complete: because $e'(m_0) > 0$, then there must exist $\hat{m}$ such that $e'(m) > 0$ for $m \in [m_0, \hat{m}]$. This means that $e(m)$ is an increasing function over

$[m_0, \hat{m}]$. To this end, first observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Therefore,

$$\lim_{m \to m_0} e'(m) = \lim_{m \to m_0} \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

L'Hôpital's rule is applicable because $\lim_{m \to m_0} k(m) = k(m_0) = 0$ and $\lim_{m \to m_0} \lambda(m) = \lambda(m_0) = 0$ hold, as we observed in the proof for Claim B.3. Thereby, we can write

$$\begin{aligned} \lim_{m \to m_0} e'(m) &= \lim_{m \to m_0} \frac{k''(m)\lambda(m) - \lambda''(m)k(m)}{2\lambda(m)\lambda'(m)} \\ &= \lim_{m \to m_0} \frac{k'''(m)\lambda(m) + k''(m)\lambda'(m) - \lambda'''(m)k(m) - \lambda''(m)k'(m)}{2\lambda'(m)^2 + 2\lambda(m)\lambda''(m)}, \end{aligned} \tag{B.5}$$

where (B.5) is due to a second application of L'Hôpital's rule. We can simplify this equality further:

$$\lim_{m \to m_0} e'(m) = \lim_{m \to m_0} \frac{k''(m)\lambda'(m)}{2\lambda'(m)^2} = \lim_{m \to m_0} \frac{k''(m)}{2\lambda'(m)} > 0, \tag{B.6}$$

where the above relation holds by Claim B.4. This proves the promised claim. □

Proposition B.9. *There exists a threshold $\hat{m}$ such that for all $m < \hat{m}$, $w'(m) > 0$.*

Proof. To prove the claim, we first note that $\lambda'(m) = F(w(m)) + mw'(m)F'(w(m))$. Hence, to prove that $\lim_{m \to m_0} w'(m) > 0$, it would suffice to show that $\lim_{m \to m_0} \lambda'(m) > 0$. This was proved by Claim B.4.

□

Proposition B.10. *There exists a threshold $\hat{m} > m_0$ such that for all $m < \hat{m}$, $(u^W)'(m) > 0$.*

Proof. We will show that $\lim_{m \to m_0^+} (u^W)'(m) > 0$. Then, continuity of $(u^W)'(m)$ at m_0 concludes the proof.

First, recall that

$$u^W(m) = \frac{1}{F(w(m))} \cdot \int_0^{w(m)} (w(m) \cdot e(m) + r \cdot (1 - e(m)) \cdot F'(r) \, \mathrm{d}\, r.$$

To compute $(u^W)'(m)$, let $B(m) = F(w(m))$, and let

$$A(m) = \int_0^{w(m)} (w(m) \cdot e(m) + r \cdot (1 - e(m)) \cdot F'(r) \, \mathrm{d}\, r.$$

Observe that

$$\lim_{m \to m_0} A(m) = \lim_{m \to m_0} B(m) = 0.$$

Hence, by L'Hôpital's rule we have

$$\begin{aligned}
\lim_{m \to m_0} (u^W)'(m) &= \lim_{m \to m_0} \frac{A'(m)B(m) - A(m)B'(m)}{B(m)^2} \\
&= \lim_{m \to m_0} \frac{A''(m)B(m) - A(m)B''(m)}{2B(m)B'(m)} \\
&= \lim_{m \to m_0} \frac{A''(m)B(m) - A(m)B''(m)}{2B(m)B'(m)} \\
&= \lim_{m \to m_0} \frac{A''(m)B(m) - A(m)B''(m)}{2B(m)B'(m)} \\
&= \frac{\lim_{m \to m_0} A''(m)B'(m) - A'(m)B''(m)}{\lim_{m \to m_0} 2(B'(m))^2}. \qquad \text{(B.7)}
\end{aligned}$$

We will show that the numerator and the denominator of (B.7) are positive, which would prove the claim. To see why the denominator is positive, observe that $B'(m) = w'(m)F'(w(m))$. By the proof of Proposition B.9, we have $\lim_{m \to m_0} w'(m) > 0$. Also, since F has a decreasing PDF, we must have that $F'(0) > 0$. Therefore, $\lim_{m \to m_0} 2(B'(m))^2 > 0$. It remains to show that the numerator of (B.7) is positive.

To this end, define

$$f(m, r) \equiv w(m)e(m) + r(1 - e(m))F'(r).$$

This is just the integrant in the definition of $A(m)$. Then, using the Leibniz integral rule, $A'(m), A''(m)$ can be computed:

$$A'(m) = f(m, w(m))w'(x) + \int_0^{w(m)} (w'(m)e(m) + w(m)e'(m) - re'(m)F'(r)) \, \mathrm{d}\, r. \qquad \text{(B.8)}$$

Let $g(m, r)$ denote the integrant in (B.8). Then, by another application of the Leibniz integral rule we have:

$$A''(m) = f(m, w(m))w''(m) + f_1(m, w(m))w'(m) + f_2(m, w(m))w'(m) + g(m, w(m))w'(m) + \int_0^{w(m)} \frac{\partial g(m,r)}{\partial m} \,\mathrm{d}\, r, \tag{B.9}$$

where $f_i(m, r)$ denotes the partial derivative of f with respect to its i-th argument.

To compute the numerator of (B.7), first note that

$$\lim_{m \to m_0} f(m, w(m)) = 0, \tag{B.10}$$
$$\lim_{m \to m_0} A'(m) = 0,$$
$$\lim_{m \to m_0} B'(m) > 0,$$
$$\lim_{m \to m_0} B''(m) = w'(m_0)^2 F''(w(m_0)) + w''(m_0)F'(w(m_0)).$$

Therefore, to show that the numerator of (B.7), i.e.,

$$\lim_{m \to m_0} A''(m)B'(m) - A'(m)B''(m).$$

is positive, it suffices to show that $\lim_{m \to m_0} A''(m)$ is positive. To this end, first note that

$$f_1(m, r) = w(m)e'(m) + w'(m)e(m) - re'(m)F'(r), \tag{B.11}$$
$$f_2(m, r) = r(1 - e(m))F''(r) + (1 - e(m))F'(r), \tag{B.12}$$
$$\lim_{m \to m_0} g(m, w(m)) = 0. \tag{B.13}$$

(B.10), (B.11), (B.12), and (B.13) together show that

$$\lim_{m \to m_0} A''(m) = F'(w(m_0)) > 0,$$

which proves the promised claim. □

C Proof of Theorem 4.1: throughput-maximizing firm

This section contains the proof for Theorem 4.1 for the case that the firm's objective is throughput maximization. The proof follows similar steps as the proof for the case of a profit-maximizing firm. For completeness, we include the full proof here. Let $S(m)$ denote the system of equations given by

$$1 - G(p + c(mF(p) - k)) - k = 0, \tag{C.1}$$

$$c'(mF(p) - k) + \frac{1}{mF(p)} = 0, \tag{C.2}$$

where (C.2) is the (simplified) firm's first-order condition with respect to p. Recall from Proposition A.5 that any nonbinding throughput-maximizing equilibrium satisfies this system. Let the function $X(m, p, k)$ and $Y(m, p, k)$ denote the LHS of equations (C.1) and (C.2), respectively. Also, let $m_0 = \frac{1}{-c'(0)F'(0)}$; we will show that $\underline{m} = m_0$.

Lemma C.1. *There exists a throughput-maximizing equilibrium at m iff $m > m_0$.*

Proof. By Claim B.2, (p, p) is feasible at m iff there exists $w < p$ such that (p, w) is feasible at m. This implies that m is feasible when the firm's objective is throughput maximizing iff it is feasible when the firm's objective is profit-maximizing. Therefore, Lemma B.1 concludes the proof. □

For the rest of the proof, it we extend the domains of the functions c, F, G so that: (i) their domains contain an interval $(-\epsilon, 0)$ for a $\epsilon > 0$, and (ii) c remains in $\mathbf{C^4}$ and strictly convex in the extended domain, (iii) F, G remain in $\mathbf{C^4}$ in the extended domain, and (iv) F' remains decreasing in the extended domain. It is straightforward to verify that such an extension exists.

Claim C.2. *There exists an open interval $I = (m_1, m_2)$ containing m_0 such that $S(m)$ has a solution at any $m \in I$. Furthermore, there exist unique continuously differentiable functions $p(m), k(m) : \mathbb{R}_+ \to \mathbb{R}$ such that $(p(m), k(m))$ is a solution to $S(m)$, for any $m \in I$.*

Proof. The proof is based on the Implicit Function Theorem. First of all, see that $(p, k) = (0, 0)$ is a solution to $S(m_0)$, i.e., $X(m_0, 0, 0) = 0$, and $Y(m_0, 0, 0) = 0$, where we recall that the functions X, Y denote the left-hand sides of the equations defining $S(m)$, i.e., the left-hand sides of (C.1) and (C.2), respectively.

To apply the implicit function theorem, we need to prove that the Jacobian

$$J(m, k, p) = \begin{pmatrix} \frac{\partial X(m,p,k)}{\partial p} & \frac{\partial X(m,p,k)}{\partial k} \\ \frac{\partial Y(m,p,k)}{\partial p} & \frac{\partial Y(m,p,k)}{\partial k} \end{pmatrix} \tag{C.3}$$

is invertible at point $(m, p, k) = (m_0, 0, 0)$. This is done in file "msm-Jacobian". There, we compute the determinant of the above matrix at point $(m_0, 0, 0)$ as

$$-m_0 c''(0)F'(0) - c''(0)G'(1) + \frac{F''(0)\left(1 - c'(0)G'(1)\right)}{m_0 F'(0)^2}.$$

Since $c''(0) > 0$, $H'(0) > 0$, $H''(0) \geq 0$, and $J'(1) \geq 0$, then the determinant is negative, and therefore the matrix is invertible at point $(m_0, 0, 0)$. Hence, the Implicit Function Theorem applies, which implies that there exist an open interval $I \ni m_0$ and unique continuously differentiable functions $p(m), k(m)$ that solve the system $S(m)$ for all $m \in I$. Furthermore, the theorem implies the existence of $p'(m)$ and $k'(m)$ for all $m \in I$.

□

Claim C.3. *Define $\lambda(m) = m \cdot F(p(m))$. Then, the following relations hold:*

$$\lim_{m \to m_0} k'(m) = 0, \qquad \lim_{m \to m_0} k''(m) > 0$$
$$\lim_{m \to m_0} \lambda'(m) > 0.$$

Furthermore, the limits

$$\lim_{m \to m_0} k'''(m), \lim_{m \to m_0} \lambda''(m), \lim_{m \to m_0} \lambda'''(m)$$

exist and are finite.

Proof. The proof follows the same steps as the proof of Claim B.3 (the counterpart to this claim but for a profit-maximizing firm). Because $p(m), k(m)$ are continuously differentiable functions (by the Implicit Function Theorem), then we have $\lim_{m \to m_0} k(m) = k(m_0)$ and $\lim_{m \to m_0} p(m) = p(m_0)$. In file "msm-poolsize", we use this fact together with implicit differentiation with respect to m from the system $S(m)$ to compute the closed-form expressions for the derivatives. We then use these closed-form expressions to compute the limits. These derivations are explained step by step in file "msm-poolsize". We go over these derivations below.

We start with computing a closed-form expression for $k'(m)$. Using implicit differentiation with respect to m from the system $S(m)$, we compute

$$k'(m) = \frac{G'(c(i) + p)\left(mF(p)F''(p)c'(i) + mF'(p)^2\left(mF(p)c''(i) - c'(i)\right) - F'(p)\right)}{m\left(mF'(p)^2 c''(i)G'(c(i) + p) + m^2 F'(p)^3 c''(i) + F''(p)\left(c'(i)G'(c(i) + p) - 1\right)\right)}$$

where on the right-hand side k, p, i denote $k(m), p(m), mF(p(m)) - k(m)$, respectively. Now, recall that $\lambda(m), p(m)$ are continuously differentiable functions. Hence, $\lim_{m\to m_0} k(m) = k(m_0)$ and $\lim_{m\to m_0} p(m) = p(m_0)$. Recall from the proof of Claim C.2 that $k(m_0) = p(m_0) = 0$. This fact and the inequalities

$$c'(0) < 0, c''(0) > 0, F'(0) > 0, F''(0) \leq 0, G'(1) > 0, \tag{C.4}$$

together with the closed-form expression for $k'(m)$ imply that $\lim_{m\to m_0} k'(m) = 0$.

We use implicit differentiation with respect to m from the system $S(m)$ to compute the closed-form expression for $k''(m)$; this is done in file "msm-poolsize" (the expression would not fit on this page). We then use $\lim_{m\to m_0} k(m) = k(m_0)$, and $\lim_{m\to m_0} \lambda(m) = \lambda(m_0)$ to compute

$$\lim_{m\to m_0} k''(m_0) = \frac{-F'(0)^2 G'(1) \left(c'(0)G'(1) - 1\right) \left(2m_0 c''(0)F'(0)^2 G'(1) + F''(0)\left(c'(0)G'(1) - 1\right) + m_0^2 c''(0)F'(0)^3 \left(c'(0)G'(1) + 1\right)\right)}{m_0 \left(m_0 F'(0) + G'(1)\right) \left(m_0 c''(0)F'(0)^2 G'(1) + m_0^2 c''(0)F'(0)^3 + F''(0)\left(c'(0)G'(1) - 1\right)\right)^2}.$$

Using (C.4) implies that the right-hand side of the above equality is positive, which proves the claim. (While the verification of the positivity of the right-hand side is straightforward from (C.4), we also show this separately in file "msm-poolsize".)

Similarly, we compute a closed-form expression for $\lim_{m\to m_0} k'''(m)$. This expression is a fraction $\frac{A}{B}$ where A is a real number (with the closed-form expression given in file "msm-poolsize") and

$$B = m_0^3 \left(m_0 c''(0)F'(0)^2 G'(1) + m_0^2 c''(0)F'(0)^3 + F''(0)\left(c'(0)G'(1) - 1\right)\right)^5.$$

By (C.4), we have $B > 0$, which implies that the limit $\lim_{m\to m_0} k'''(m)$ exists.

It remains to compute the limits regarding $\lambda(m)$. In file "msm-poolsize" we use implicit differentiation with respect to m from the system $S(m)$ to compute

$$\lim_{m\to m_0} \lambda'(m) = \frac{-F'(0)\left(c'(0)G'(1) - 1\right)}{m_0 c''(0)F'(0)^2 G'(1) + m_0^2 c''(0)F'(0)^3 + F''(0)\left(c'(0)G'(1) - 1\right)}.$$

By (C.4), the right-hand side is positive.

We also compute the closed-form expression for $\lim_{m\to m_0} \lambda''(m)$ as $\frac{C}{D}$ where C is a real number and

$$D = m_0 \left(m_0 c''(0) F'(0)^2 G'(1) + m_0^2 c''(0) F'(0)^3 + F''(0) \left(c'(0) G'(1) - 1 \right) \right)^3 .$$

By (C.4), $D > 0$, which implies that the limit $\lim_{m \to m_0} k'''(m)$ exists.

Finally, we compute the closed-form expression for $\lim_{m \to m_0} \lambda'''(m)$ as $\frac{E}{F}$ where E is a real number and

$$F = m_0^2 \left(m_0 c''(0) F'(0)^2 G'(1) + m_0^2 c''(0) F'(0)^3 + F''(0) \left(c'(0) G'(1) - 1 \right) \right)^5 .$$

By (C.4), we have that $F > 0$, which implies that the limit $\lim_{m \to m_0} \lambda'''(m)$ exists. This concludes the proof.

□

Claim C.4. *There exists $m_3 > m_0$ such that $\lambda'(m) \neq 0$ for any $m \in (m_0, m_3)$.*

Proof. This is a consequence of Claim C.3 and the continuity of $\lambda'(m)$. (We recall that the continuity of $\lambda'(m)$ is granted by the Implicit Function Theorem). □

To prove the claim of the theorem, we will show that there exists $\hat{m} > m_0$ such that $e'(m) > 0$, $w'(m) > 0$, and $(u^W)'(m)$ hold for all $m \in (m_0, \hat{m})$. This will be shown in Proposition C.5, Proposition C.8, and Proposition C.9.

Proposition C.5. *There exists a threshold $\hat{m}$ such that for all $m \in (m_0, \hat{m})$, $e'(m) > 0$.*

Proof. First, we prove the following claims.

Claim C.6. *There exists $m_4 > m_0$ such that $e'(m)$ exists at all $m \in (m_0, m_1)$.*

Proof. First, observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Because $\lambda(m) > 0$ for all $m > m_0$, and because $k(m), k'(m)$ and $\lambda'(m)$ exist and are finite for m sufficiently close to m_0 (by Claim C.3), $e'(m)$ exists and is finite for m sufficiently close to m_0.

□

Claim C.7. $\lim_{m \to m_0} e(m) = 0$.

Proof. First, observe that

$$\lim_{m \to m_0} e(m) = \lim_{m \to m_0} \frac{k(m)}{\lambda(m)} = \lim_{m \to m_0} \frac{k'(m)}{\lambda'(m)}.$$

L'Hôpital's rule is applicable here by Claim C.4. By Claim C.3, $\lim_{m \to m_0} k'(m) = 0$ and $\lim_{m \to m_0} \lambda'(m) > 0$, which concludes the proof. $\square$

Continuity of $e(m)$ at m_0 is ensured by Claim C.7. The rest of the proof is as follows. We will show that $\lim_{m \to m_0} e'(m)$ exists and is positive. This would imply that $e'(m_0)$ must also exist, and in fact that

$$e'(m_0) = \lim_{m \to m_0} e'(m).$$

(This is a consequence of L'Hôpital's rule. See, for example, [Wikipedia 2017] for a proof.) Once we have shown this, the proof is complete: because $e'(m_0) > 0$, then there must exist $\hat{m}$ such that $e'(m) > 0$ for $m \in [m_0, \hat{m}]$, i.e., $e(m)$ is an increasing function over $[m_0, \hat{m}]$.

First, observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Therefore,

$$\lim_{m \to m_0} e'(m) = \lim_{m \to m_0} \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

L'Hôpital's rule is applicable because $\lim_{m \to m_0} k(m) = k(m_0) = 0$ and $\lim_{m \to m_0} \lambda(m) = \lambda(m_0) = 0$ hold, as we observed in Claim C.2 and its proof. Thereby, we can write

$$\begin{aligned} \lim_{m \to m_0} e'(m) &= \lim_{m \to m_0} \frac{k''(m)\lambda(m) - \lambda''(m)k(m)}{2\lambda(m)\lambda'(m)} \\ &= \lim_{m \to m_0} \frac{k'''(m)\lambda(m) + k''(m)\lambda'(m) - \lambda'''(m)k(m) - \lambda''(m)k'(m)}{2\lambda'(m)^2 + 2\lambda(m)\lambda''(m)}, \end{aligned} \tag{C.5}$$

where (C.5) is due to a second application of L'Hôpital's rule. We can simplify this equality further and write

$$\lim_{m \to m_0} e'(m) = \lim_{m \to m_0} \frac{k''(m)\lambda'(m)}{2\lambda'(m)^2} = \lim_{m \to m_0} \frac{k''(m)}{2\lambda'(m)} > 0, \tag{C.6}$$

where the above relation holds by Claim C.3. This proves the promised claim. $\square$

Proposition C.8. *There exists $\hat{m} > m_0$ such that for all $m \in [m_0, \hat{m})$, $p'(m) > 0$.*

Proof. We show that $\lim_{m \to m_0^+} p'(m) > 0$ in file "msm-poolsize". To do this, we first use implicit differentiation with respect to m from the system $S(m)$ to compute a closed-form expression for $p'(m)$ as follows:

$$p'(m) = -\frac{m^2 F(p) F'(p)^2 c''(i) + F'(p) c'(i) G'(c(i) + p) - F'(p)}{m \left(m F'(p)^2 c''(i) G'(c(i) + p) + m^2 F'(p)^3 c''(i) + F''(p) c'(i) G'(c(i) + p) - F''(p) \right)},$$

where on the right-hand side p, i respectively denote $p(m), mF(p(m)) - k(m)$. Using this closed-form expression and the facts that $\lim_{m \to m_0} p(m) = p(m_0) = 0$ and $\lim_{m \to m_0} k(m) = k(m_0) = 0$, we can compute

$$\lim_{m \to m_0^+} p'(m) = -\frac{F'(0) \left(c'(0) G'(1) - 1 \right)}{m \left(m c''(0) F'(0)^2 G'(1) + m^2 c''(0) F'(0)^3 + F''(0) \left(c'(0) G'(1) - 1 \right) \right)}.$$

By (C.4), the right-hand side of the above equality is positive. Continuity of $p'(m)$ then implies that there exists $\delta > 0$ such that $p'(m) > 0$ for $m \in [m_0, m_0 + \delta]$. This proves the claim. □

Proposition C.9. *There exists a threshold $\hat{m} > m_0$ such that for all $m < \hat{m}$, $(u^W)'(m) > 0$.*

Proof. The proof is identical to the proof of Proposition B.10. □

D Proof of Theorem 5.1: profit-maximizing firm

This section contains the proof for Theorem 5.1 for the case that the firm's objective is profit maximization. The proof structure is similar to the proof of Theorem 4.1. First, we need some notation. We use $c_1 : \mathbb{R}_+^2 \to \mathbb{R}$ and $c_2 : \mathbb{R}+^2 \to \mathbb{R}$ to denote the partials of c with respect to its first and second argument, respectively.

Any monopoly equilibrium must satisfy the following equations

$$k \cdot \left(c_2(\gamma, \lambda - k) - J'(1 - k) \right) - c(\gamma, \lambda - k) - H\left(\lambda / m \right) + J(1 - k) = 0, \tag{D.1}$$

$$c_2(\gamma, \lambda - k) + \frac{H'\left(\lambda / m \right)}{m} = 0. \tag{D.2}$$

Note that these equations are written for a fixed level of matching technology. They are the same as equations (A.3) and (A.4), written using the new notation for the cost function. We use the notation $S(m, \gamma)$ to refer to the above system of equations. The next lemma shows that $\underline{m}_\gamma = \frac{1}{-c_2(\gamma, 0) F'(0)}$.

Lemma D.1. *For any γ, there exists a monopoly equilibrium at m iff $m > \frac{1}{-c_2(\gamma,0)F'(0)}$.*

Proof. The proof is identical to the proof of Lemma B.1, since the lemma concerns a fixed level of matching technology. $\square$

For the rest of the proof, we fix a level of matching technology γ_0 and prove the theorem statement for γ_0. For notational simplicity, let $m_0 = \underline{m}_{\gamma_0}$. Throughout the rest of the proof, we assume that $c_{1,2}(\gamma_0, m_0) \neq 0$, as assumed in the theorem statement.

Next, we present two counterparts for Claim B.3 and Claim B.4 in the proof of Theorem 4.1.

Claim D.2. *There exist open intervals $I = (m_1, m_2)$ and $I' = (\gamma_1, \gamma_2)$ with $m_0 \in I$ and $\gamma_0 \in I'$ such that $S(m, \gamma)$ has a solution for any m, γ with $m \in I$ and $\gamma \in I'$. Furthermore, there exist unique continuously differentiable functions $\lambda(m, \gamma), k(m, \gamma) : \mathbb{R}_+ \to \mathbb{R}$ such that $(\lambda(m, \gamma), k(m, \gamma))$ is a solution to $S(m, \gamma)$, for any m, γ with $m \in I$ and $\gamma \in I'$.*

Proof. The proof is similar to the proof of Claim B.3. We include the proof for completeness.

The proof is based on the Implicit Function Theorem. As we mentioned earlier, we apply the theorem to the system given by (D.1) and (D.2). First, recall that (D.1) and (D.2) are respectively given by

$$k \cdot (c_2(\gamma, \lambda - k) - J'(1 - k)) - c(\gamma, \lambda - k) - H(\lambda/m) + J(1 - k) = 0,$$
$$c_2(\gamma, \lambda - k) + \frac{H'(\lambda/m)}{m} = 0.$$

We denote the left-hand sides of (D.1) and (D.2) respectively by $X(m, \lambda, k, \gamma)$ and $Y(m, \lambda, k, \gamma)$. Next, we show that $(\lambda, k) = (0, 0)$ is a solution to $S(m_0, \gamma_0)$, i.e., $X(m_0, 0, 0, \gamma_0) = 0$, and $Y(m_0, 0, 0, \gamma_0) = 0$, as follows:

$$0 \cdot (c_2(\gamma, 0) - J'(1)) - c(\gamma, 0) - H(0) + J(1) = 0 - 1 - 0 + 1 = 0$$
$$c_2(\gamma, 0) + \frac{H'(0)}{m_0} = 0,$$

where the last equality holds because $m_0 = \frac{-1}{F'(0)c_2(\gamma,0)}$ and $H'(0) = \frac{1}{F'(0)}$.

To apply the implicit function theorem, we need to prove that the Jacobian

$$J(m, k, \lambda, \gamma) = \begin{pmatrix} \frac{\partial X(m,\lambda,k,\gamma)}{\partial \lambda} & \frac{\partial X(m,\lambda,k,\gamma)}{\partial k} \\ \frac{\partial Y(m,\lambda,k,\gamma)}{\partial \lambda} & \frac{\partial Y(m,\lambda,k,\gamma)}{\partial k} \end{pmatrix}$$

is invertible at point $(m, \lambda, k, \gamma) = (m_0, 0, 0, \gamma_0)$. Observe that this Jacobian involves only a fixed level of matching technology, γ_0. Define the function $c(\cdot) = c(\gamma_0, \cdot)$, and then observe that the above Jacobian is identical to (B.2), under this notation. The proof of Claim B.3 shows that this Jacobian is invertible at point $(m_0, 0, 0, \gamma_0)$. Hence, the Implicit Function Theorem therefore applies, and there exists open intervals $I \ni m_0$ and $I' \ni \gamma_0$ and unique continuously differentiable functions $\lambda(m, \gamma), k(m, \gamma)$ that solve the system $S(m, \gamma)$ for all $m \in I$ and $\gamma \in I'$. Furthermore, the theorem implies that the functions $\lambda(\cdot, \cdot)$ and $k(\cdot, \cdot)$ are continuously differentiable over $I \times I'$.

□

We use the notation $\lambda_i(m, \gamma), k_i(m, \gamma)$ to denote the partials of the functions λ, k with respect to their i-th argument, for $i \in \{1, 2\}$. Furthermore, $k_{i,j}(m, \gamma)$ denotes the cross-partial with respect to the i-th and j-th arguments (with the possibly that $i = j$). Also define $k_{i,j,k}(m, \gamma)$ similarly. The existence of such (cross-)partials is always proved when we refer to one in a formal statement.

Claim D.3. *The following relations hold:*

$$\begin{aligned} & k_2(m_0, \gamma_0) = 0, \quad k_{2,2}(m_0, \gamma_0) > 0, \\ & \lambda_2(m_0, \gamma_0) > 0. \end{aligned}$$

Furthermore, the limits

$$\lim_{m \to m_0} k_{2,2,2}(m, \gamma_0), \lim_{m \to m_0} \lambda_{2,2}(m, \gamma_0), \lim_{m \to m_0} \lambda_{2,2,2}(m, \gamma_0)$$

exist and are finite.

Proof. We prove the claim by implicit differentiation with respect to γ from the system $S(m, \gamma)$ to compute the closed-form expressions for the derivatives and their limits. The step-by-step derivations are done in file "MT". We go over these derivations below.

First, we compute

$$k_2(m,\gamma) = \frac{-\left(\frac{H''\left(\frac{\lambda}{m}\right)(kc_{1,2}(\gamma,i)-c_1(\gamma,i))}{m^2} - c_{2,2}(\gamma,i)c_1(\gamma,i)\right)}{\left(c_{2,2}(\gamma,i)+\frac{H''\left(\frac{\lambda}{m}\right)}{m^2}\right)\left(k\left(J''(1-k)-c^{(0,2)}(\gamma,i)\right)-\frac{2H'\left(\frac{\lambda}{m}\right)}{m}-2J'(1-k)\right)+kc_{2,2}(\gamma,i)^2}, \tag{D.3}$$

where on the right-hand side k denotes $k(m,\gamma)$, λ denotes $\lambda(m,\gamma)$, i denotes $\lambda - k$, $c_j(\cdot,\cdot)$ denotes the partial of c with respect to its j-th argument, and $c_{j,l}(\cdot,\cdot)$ denotes the cross-partial of c with respect to its j-th and l-th arguments (with the possibly that $j = l$). The facts that

$$k(m_0,\gamma_0) = 0, \lambda(m_0,\gamma_0) = 0, c_1(\gamma_0, 0) = 0$$

imply that the numerator of (D.3) is 0 when $m = m_0$ and $\gamma = \gamma_0$. The facts that

$$c_{2,2}(\gamma_0, 0) > 0, H'(0) > 0, H''(0) \geq 0, J'(1) > 0 \tag{D.4}$$

imply that the denominator of (D.3) is nonzero when $m = m_0$ and $\gamma = \gamma_0$. This proves that $k_2(m_0,\gamma_0) = 0$.

To prove that $k_{2,2}(m_0,\gamma_0) > 0$, we compute the closed-form expression for $k_{2,2}(m,\gamma)$ by implicit differentiation with respect to γ from the system $S(m,\gamma)$. This is done in file "MT". We then use the closed-form expression to compute

$$k_{2,2}(m_0,\gamma_0) = \frac{m_0^3 c_{1,2}(\gamma_0,0)^2}{2\left(H'(0)+m_0 J'(1)\right)\left(m_0^2 c_{2,2}(\gamma_0,0)+H''(0)\right)}.$$

The fact that $c_{1,2}(\gamma_0, 0) \neq 0$ implies that the numerator is positive, and (D.4) implies that the denominator is positive. This proves that $k_{2,2}(m_0,\gamma_0) > 0$.

We prove that the limit $\lim_{m\to m_0} k_{2,2,2}(m,\gamma_0)$ exists, as follows. We compute the closed-form expression for $k_{2,2,2}(m,\gamma)$ and show that when $\gamma = \gamma_0$, the closed-form expression approaches a fraction $\frac{A}{B}$ as m approaches m_0, where A is a real number and

$$B = 2\left(H'(0)+m_0 J'(1)\right)^2\left(m_0^2 c_{2,2}(\gamma_0,0)+H''(0)\right)^3.$$

Observe that, by (D.4), $B > 0$, which shows that $\lim_{m\to m_0} k_{2,2,2}(m,\gamma_0)$ exists.

To prove that $\lambda_2(m_0,\gamma_0) > 0$, we compute the closed-form expression for $\lambda_2(m_0,\gamma_0)$ by implicit differentiation with respect to γ from the system $S(m,\gamma)$. This is done in file "MT". We then use the closed-form expression to compute

$$\lambda_2(m_0,\gamma_0) = -\frac{m_0^2 c_{1,2}(\gamma_0,0)}{m_0^2 c_{2,2}(\gamma_0,0) + H''(0)}.$$

The fact that $c_{1,2}(\gamma_0,0) \neq 0$ and regularity of $c(\gamma,\cdot)$ for every γ imply that $c_{1,2}(\gamma_0,0) < 0$. This, together with (D.4), implies that the right-hand side of the above equation is positive, which proves the claim.

We prove that the limit $\lim_{m\to m_0} \lambda_{2,2}(m,\gamma_0)$ exists, as follows. We compute the closed-form expression for $\lambda_{2,2}(m,\gamma)$, and show that when $\gamma = \gamma_0$, the closed-form expression approaches a fraction $\frac{C}{D}$ as m approaches m_0, where C is a real number and

$$D = 2\left(m_0^2 c_{2,2}(\gamma_0,0) + H''(0)\right)^3.$$

Since $D > 0$ holds by (D.4), then $\lim_{m\to m_0} \lambda_{2,2}(m,\gamma_0)$ exists.

The proof for the existence of the limit $\lim_{m\to m_0} \lambda_{2,2,2}(m,\gamma_0)$ is similar, as follows. We compute the closed-form expression for $\lambda_{2,2,2}(m,\gamma)$ and show that when $\gamma = \gamma_0$, the closed-form expression approaches a fraction $\frac{E}{F}$ as m approaches m_0, where E is a real number and

$$F = 2\left(H'(0) + m_0 J'(1)\right)^2 \left(m_0^2 c_{2,2}(\gamma_0,0) + H''(0)\right)^5.$$

Since $F > 0$ holds by (D.4), then $\lim_{m\to m_0} \lambda_{2,2,2}(m,\gamma_0)$ exists. This concludes the proof. □

To prove Theorem 5.1, we will show that there exists $\hat{m} > m_0$ such that $e_2(m,\gamma_0) > 0$, $w_2(m,\gamma_0) > 0$, and $(u^W)_2(m,\gamma_0) > 0$ hold for all $m \in (m_0,\hat{m})$. This will be shown in Proposition D.4, Proposition D.5, and Proposition D.6.

Proposition D.4. *There exists a threshold $\hat{m}$ such that for all $m \in (m_0,\hat{m})$, $e_2(m,\gamma_0) > 0$.*

Proof. First, observe that for all $m > m_0$ we have

$$e_2(m,\gamma_0) = \frac{k_2(m,\gamma_0)\lambda(m,\gamma_0) - \lambda_2(m,\gamma_0)k(m,\gamma_0)}{\lambda(m,\gamma_0)^2}.$$

Therefore,

$$\lim_{m\to m_0} e_2(m,\gamma_0) = \lim_{m\to m_0} \frac{k_2(m,\gamma_0)\lambda(m,\gamma_0) - \lambda_2(m,\gamma_0)k(m,\gamma_0)}{\lambda(m,\gamma_0)^2}.$$

L'Hôpital's rule is applicable because the following hold: $\lim_{m\to m_0} k(m,\gamma_0) = k(m_0,\gamma_0) = 0$ and $\lim_{m\to m_0} \lambda(m,\gamma_0) = \lambda(m_0,\gamma_0) = 0$. (The proof is similar to the proof of Claim B.3.) Thereby, we can write

$$\begin{aligned} &\lim_{m\to m_0} e_2(m,\gamma_0) && \text{(D.5)} \\ &= \lim_{m\to m_0} \frac{k_{2,2}(m,\gamma_0)\lambda(m,\gamma_0) - \lambda_{2,2}(m,\gamma_0)k(m)}{2\lambda(m,\gamma_0)\lambda_2(m,\gamma_0)} \\ &= \lim_{m\to m_0} \frac{k_{2,2,2}(m,\gamma_0)\lambda(m,\gamma_0) + k_{2,2}(m,\gamma_0)\lambda_2(m,\gamma_0) - \lambda_{2,2,2}(m,\gamma_0)k(m,\gamma_0) - \lambda_{2,2}(m,\gamma_0)k_2(m,\gamma_0)}{2\lambda_2(m,\gamma_0)^2 + 2\lambda(m,\gamma_0)\lambda_{2,2}(m,\gamma_0)}, \\ & && \text{(D.6)} \end{aligned}$$

where (D.6) is due to a second application of L'Hôpital's rule. We can simplify this equality further and write

$$\lim_{m\to m_0} e_2(m,\gamma_0) = \lim_{m\to m_0} \frac{k_{2,2}(m,\gamma_0)\lambda_2(m,\gamma_0)}{2\lambda_2(m,\gamma_0)^2} = \lim_{m\to m_0} \frac{k_{2,2}(m,\gamma_0)}{2\lambda_2(m,\gamma_0)} > 0, \tag{D.7}$$

where the above relation holds by Claim D.3. This proves the promised claim. □

Proposition D.5. *There exists a threshold $\hat{m}$ such that for all $m < \hat{m}$, $w_2(m,\gamma_0) > 0$.*

Proof. To prove the claim, we first note that $\lambda_2(m,\gamma) = mw_2(m,\gamma)F'(w(m,\gamma))$. Hence, to prove that $\lim_{m\to m_0} w_2(m,\gamma) > 0$, it would suffice to show that $\lim_{m\to m_0} \lambda_2(m,\gamma) > 0$. This was shown in Claim D.3. Continuity of the function $\lambda_2(\cdot,\cdot)$ then implies that there exists $\delta > 0$ such that $\lambda_2(m,\gamma_0) > 0$ for $m \in [m_0, m_0+\delta]$. Setting $\hat{m} = m_0+\delta$ proves the claim. □

Proposition D.6. *There exists a threshold $\hat{m} > m_0$ such that for all $m < \hat{m}$, $(u^W)'(m) > 0$.*

Proof. The proof is essentially identical to the proof of Proposition B.10. □

E Proof of Theorem 5.1: throughput-maximizing firm

This section contains the proof for Theorem 5.1 for the case that the firm's objective is throughput maximization. The proof structure is similar to the proof of Theorem 4.1 for the case that the firm's objective is throughput maximization. First, we need some notation. We use $c_1 : \mathbb{R}_+^2 \to \mathbb{R}$ and $c_2 : \mathbb{R}+^2 \to \mathbb{R}$ to denote the partials of c with respect to its first and second argument, respectively.

For a fixed γ, Proposition A.5 shows that any throughput-maximizing equilibrium must satisfy the following equations:

$$1 - G(p + c(\gamma, fmF(p) - k)) - k = 0, \tag{E.1}$$

$$c_2(\gamma, mF(p) - k) + \frac{1}{mF(p)} = 0. \tag{E.2}$$

We use the notation $S(m, \gamma)$ to refer to the above system of equations and let $X(m, \gamma, p, k)$ and $Y(m, \gamma, p, k)$ respectively denote the right-hand sides of (E.1) and (E.2).

The next lemma shows that $\underline{m}_\gamma = \frac{1}{-c_2(\gamma,0)F'(0)}$.

Lemma E.1. *For any γ, there exists a throughput-maximizing equilibrium at m iff $m > \frac{1}{-c_2(\gamma,0)F'(0)}$.*

Proof. Since the lemma is concerned with a fixed level of matching technology, its proof is the same as the proof of Lemma C.1. More precisely, replacing $c(\cdot)$ with $c(\gamma, \cdot)$ in the proof of Lemma C.1 gives a proof for this lemma. □

For the rest of the proof, we fix a level of matching technology γ_0 and prove the theorem statement for γ_0. For notational simplicity, let $m_0 = \underline{m}_{\gamma_0}$. Throughout the rest of the proof, we assume that $c_{1,2}(\gamma_0, m_0) \neq 0$, as assumed in the theorem statement.

Next, we present two counterparts for Claim C.2 and Claim C.3 which were used in the proof for the profit-maximizing share.

Claim E.2. *There exist open intervals $I = (m_1, m_2)$ and $I' = (\gamma_1, \gamma_2)$ with $m_0 \in I$ and $\gamma_0 \in I'$ such that $S(m, \gamma)$ has a solution for any m, γ with $m \in I$ and $\gamma \in I'$. Furthermore, there exist unique continuously differentiable functions $p(m, \gamma), k(m, \gamma) : \mathbb{R}_+ \to \mathbb{R}$ such that $(p(m, \gamma), k(m, \gamma))$ is a solution to $S(m, \gamma)$, for any m, γ with $m \in I$ and $\gamma \in I'$.*

Proof. The proof for Claim E.2 is essentially identical to the proof for Claim C.2; the proof applies the Implicit Function Theorem. To apply the theorem, we need to show that the Jacobian

$$J(m, \gamma, k, p) = \begin{pmatrix} \frac{\partial X(m,\gamma,p,k)}{\partial p} & \frac{\partial X(m,\gamma,p,k)}{\partial k} \\ \frac{\partial Y(m,\gamma,p,k)}{\partial p} & \frac{\partial Y(m,\gamma,p,k)}{\partial k} \end{pmatrix} \tag{E.3}$$

is invertible at point $(m, \gamma, k, p) = (m_0, \gamma_0, 0, 0)$. We have already shown this in the proof of Theorem 4.1 for the case of the throughput-maximizing firm: observe that denoting the function $c(\gamma_0, \cdot)$ by $c(\cdot)$ makes (E.3) identical to the Jacobian (C.3) in the proof of Theorem 4.1. □

We use the notation $\lambda_i(m, \gamma), k_i(m, \gamma)$ to denote the partials of the functions λ, k with respect to their i-th argument, for $i \in \{1, 2\}$. Furthermore, $k_{i,j}(m, \gamma)$ denotes the cross-partial derivative with respect to the i-th and j-th arguments (with the possibility of $i = j$). Also define $k_{i,j,k}(m, \gamma)$ similarly.

Claim E.3. *Let $\lambda(m, \gamma) = m \cdot F(p(m, \gamma))$. Then,*

$$\begin{aligned}
&k_2(m_0, \gamma_0) = 0, \quad k_{2,2}(m_0, \gamma_0) > 0, \\
&\lambda_2(m_0, \gamma_0) > 0.
\end{aligned}$$

Furthermore, the limits

$$\lim_{m \to m_0} k_{2,2,2}(m, \gamma_0), \lim_{m \to m_0} \lambda_{2,2}(m, \gamma_0), \lim_{m \to m_0} \lambda_{2,2,2}(m, \gamma_0)$$

exist and are finite.

Proof. We prove the claim by implicit differentiation with respect to m from the system $S(m, \gamma)$ to compute the closed-form expressions for the derivatives and their limits. The step-by-step derivations are done in file "msm-MT". We go over these derivations below.

We prove that $k_2(m_0, \gamma_0) = 0$ as follows. We compute the closed-form expression for $k_2(m, \gamma)$ as

$$\frac{G'(c(\gamma, i) + p)\left(F''(p)c_1(\gamma, i) + mF'(p)^2\left(F'(p)\left(mc_2(\gamma, i)c_{1,2}(\gamma, i) - mc_{2,2}(\gamma, i)c_1(\gamma, i)\right) + c_{1,2}(\gamma, i)\right)\right)}{F''(p)\left(c_2(\gamma, i)G'(c(\gamma, i) + p) - 1\right) + mF'(p)^2 c_{2,2}(\gamma, i)\left(G'(c(\gamma, i) + p) + mF'(p)\right)} \tag{E.4}$$

where k denotes $k(m, \gamma)$, p denotes $p(m, \gamma)$, i denotes $mF(p) - k$, $c_j(\cdot, \cdot)$ denotes the partial of c with respect to its j-th argument, and $c_{j,l}(\cdot, \cdot)$ denotes the cross-partial of c with respect to its j-th and l-th arguments (with the possibility of $j = l$). The facts

$$\begin{cases} p(m_0, \gamma_0) = 0, k(m_0, \gamma_0) = 0, \\ F'(0) > 0, F''(0) \leq 0, G'(1) > 0, \\ c_1(\gamma_0, 0) = 0, c_2(\gamma_0, 0) = \frac{-1}{m_0 F'(0)}, c_{2,2}(\gamma_0, 0) > 0, \end{cases} \tag{E.5}$$

imply that the denominator of (E.4) is positive and that its numerator is 0. This means that $k_2(m, \gamma_0) = 0$.

To prove that $k_{2,2}(m_0, \gamma_0) > 0$, we compute a closed-form expression for it by implicit differentiation with respect to γ from the system $S(m, \gamma)$. This is done in file "msm-MT". We then use the closed-form expression to compute

$$k_{2,2}(m_0, \gamma_0) = \frac{F'(0)G'(1)c_{1,2}(\gamma_0, 0)^2 \left(m_0 F'(0) + G'(1)\right)^2 \left(F''(0) - m_0^2 F'(0)^3 c_{2,2}(\gamma_0, 0)\right)^2}{\left(F''(0)G'(1)c_2(\gamma_0, 0) + m_0 F'(0)^2 c_{2,2}(\gamma_0, 0) \left(m_0 F'(0) + G'(1)\right) - F''(0)\right)^3}.$$

The fact that $c_{1,2}(\gamma_0, 0) \neq 0$ and (E.5) together imply that the numerator is positive. Also, (E.5) implies that the denominator is positive. This proves that $k_{2,2}(m_0, \gamma_0) > 0$.

We prove that the limit $\lim_{m \to m_0} k_{2,2,2}(m, \gamma_0)$ exists, as follows. We compute the closed-form expression for $k_{2,2,2}(m, \gamma)$ and show that when $\gamma = \gamma_0$, the closed-form expression approaches a fraction $\frac{A}{B}$ as m approaches m_0, where A is a real number and

$$B = \left(F''(0)G'(1)c_2(\gamma_0, 0) + m_0 F'(0)^2 c_{2,2}(\gamma_0, 0) \left(m_0 F'(0) + G'(1)\right) - F''(0)\right)^5.$$

Observe that, by (E.5), $B > 0$, which shows that $\lim_{m \to m_0} k_{2,2,2}(m, \gamma_0)$ exists.

To prove that $\lambda_2(m_0, \gamma_0) > 0$, we compute the closed-form expression for $\lambda_2(m_0, \gamma_0)$ by implicit differentiation with respect to m from the system $S(m, \gamma)$. We then use the closed-form expression to compute

$$\lambda_2(m_0, \gamma_0) = \frac{-m_0 F'(0) c_{1,2}(\gamma_0, 0) \left(m_0 F'(0) + G'(1)\right)}{F''(0)G'(1)c_2(\gamma_0, 0) + m_0 F'(0)^2 c_{2,2}(\gamma_0, 0) \left(m_0 F'(0) + G'(1)\right) - F''(0)}.$$

The fact that $c_{1,2}(\gamma_0, 0) \neq 0$ and the regularity of $c(\gamma, \cdot)$ for every γ imply that $c_{1,2}(\gamma_0, 0) < 0$. This, together with (E.5), implies that the right-hand side of the above equation is positive, which proves the claim.

We prove that the limit $\lim_{m \to m_0} \lambda_{2,2}(m, \gamma_0)$ exists, as follows. We compute the closed-form expression for $\lambda_{2,2}(m, \gamma)$, and show that when $\gamma = \gamma_0$, the closed-form expression approaches a fraction $\frac{C}{D}$ as m approaches m_0, where C is a real number and

$$D = \left(F''(0)G'(1)c_2(\gamma_0,0) + m_0F'(0)^2c_{2,2}(\gamma_0,0)\left(m_0F'(0)+G'(1)\right) - F''(0)\right)^2.$$

Since $D > 0$ holds by (E.5), then $\lim_{m\to m_0}\lambda_{2,2}(m,\gamma_0)$ exists.

The proof for the existence of the limit $\lim_{m\to m_0}\lambda_{2,2,2}(m,\gamma_0)$ is similar, as follows. We compute the closed-form expression for $\lambda_{2,2,2}(m,\gamma)$ and show that when $\gamma=\gamma_0$, the closed-form expression approaches a fraction $\frac{E}{F}$ as m approaches m_0, where E is a real number and

$$F = \left(F''(0)G'(1)c_2(\gamma_0,0) + m_0F'(0)^2c_{2,2}(\gamma_0,0)\left(m_0F'(0)+G'(1)\right) - F''(0)\right)^3.$$

Since $F > 0$ holds by (E.5), then $\lim_{m\to m_0}\lambda_{2,2,2}(m,\gamma_0)$ exists. This concludes the proof. □

To prove the theorem, we will show that there exists $\hat{m} > m_0$ such that $e_2(m,\gamma_0) > 0$, $w_2(m,\gamma_0) > 0$, and $(u^W)'(m) > 0$ hold for all $m \in (m_0,\hat{m})$. This will be shown in Proposition E.4, Proposition E.5, and Proposition E.6.

Proposition E.4. *There exists a threshold $\hat{m}$ such that for all $m \in (m_0,\hat{m})$, $e_2(m,\gamma_0) > 0$.*

Proof. The proof is identical to the proof for Proposition D.4, except that now we use Claim E.3 instead of Claim D.3 in the proof of that proposition. □

Proposition E.5. *There exists a threshold $\hat{m}$ such that for all $m < \hat{m}$, $p_2(m,\gamma_0) > 0$.*

Proof. Claim E.3 shows that $\lambda_2(m_0,\gamma_0) > 0$, where recall that $\lambda(m,\gamma) = mF(p(m,\gamma))$, by definition. Hence, $p_2(m_0,\gamma_0) > 0$. Continuity of the function $p_2(\cdot,\cdot)$ then implies that there exists $\delta > 0$ such that $p_2(m,\gamma_0) > 0$ for $m \in [m_0, m_0+\delta]$. This proves the claim. □

Proposition E.6. *There exists a threshold $\hat{m} > m_0$ such that for all $m < \hat{m}$, $(u^W)'(m) > 0$.*

Proof. The proof is essentially identical to the proof of Proposition B.10. □

F Preliminary results on duopoly

F.1 Notation

When referring to the customer composition of a subgame equilibrium $\Sigma = (\mathbf{P},\mathbf{A})$, we use the variable $\mathbf{k} = (k_1,k_2)$, unless otherwise specified. Furthermore, we use the variable k to denote $k_1 + k_2$. We also use the variables p_f, w_f to denote the price and wage at firm f in the payment profile $\mathbf{P}$.

Given Σ, we also define the notion of the *aggregate cost* that customers face at firm f to be $p_f + c(i_f)$, where i_f is the number of idle workers who accept offers from firm f in Σ. Unless otherwise specified, we use the variable b_f to denote $p_f + c(i_f)$.

In the analysis we sometimes consider a variable other than Σ, typically Σ', to denote a subgame equilibrium. In that case, we will use a notation similar to the above notation to refer to the parameters in Σ'. For example, p'_f, w'_f will denote price and wage at firm f and k'_f will denote the rate of customers who join firm f in Σ'.

The firm's demand function The function $D : [0,1]^2 \to [0,1]$ is the *demand function* of customers. $D(b_1, b_2)$ determines the mass of customers demanding to join firm 1, assuming that the aggregate cost at firm f is b_f. By the symmetry of our model, $D(b_2, b_1)$ is the mass of customers demanding to join firm 2. This function has a simple geometric representation, depicted in Figure 10. Each point (x, y) in the unit square represents a customer with valuation (v_1, v_2) defined as

$$\begin{aligned} v_1 &= \sigma x + (1-\sigma)y, \\ v_2 &= \sigma x + (1-\sigma)(1-y). \end{aligned}$$

Line l_1 corresponds to the customers who earn 0 payoff from joining firm 1. More precisely, l_1 is the line $\sigma x + (1-\sigma)y = b_1$. Similarly, line l_2 corresponds to the customers who earn 0 payoff from joining firm l_2, i.e., the line $\sigma x + (1-\sigma)(1-y) = b_2$. The red shaded area is $D(b_1, b_2)$, and the blue shaded area is $D(b_2, b_1)$. The function D, obviously, has a closed-form expression in terms of b_1, b_2.

For notational simplicity, we sometimes denote $1 - \sigma$ by a throughout the analysis.

F.2 Preliminaries

Definition F.1. *A payment profile $((p_1, w_1), (p_2, w_2))$ is called a standard payment profile if it induces a nontrivial subgame equilibrium Σ such that $c'(mw_1 - k) = c'(mw_2 - k) = \frac{-1}{m}$, where k is the total rate of customers requesting service in Σ.*

Lemma F.2. *In any nontrivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, there exists a cutoff r^* such that all workers with $r < r^*$ choose to accept offers from both firms and all workers with*

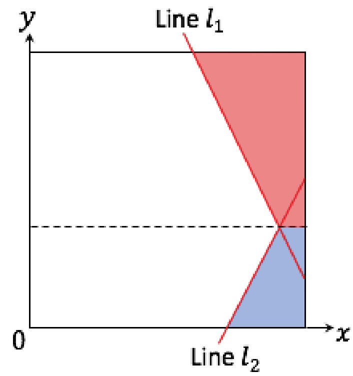

Figure 10: A graphical representation for the demand function

$r \in (r^*, \max\{w_1, w_2\}]$ *choose to accept offers only from the firm with the maximum wage. If case* $w_1 = w_2$, *then* $r^* = w_1$.

Proof. If $w_1 = w_2$, the claim is proved, because any individual worker who joins only one firm could strictly increase her payoff by joining both firms. Hence, suppose without loss of generality that $w_1 > w_2$. Any worker who accepts offers from firm 2 in Σ should also accept offers from firm 1; if not, she can increase her payoff by accepting offers from firm 1 as well. Therefore, the set of workers could be partitioned into 3 subsets: workers of the *high* type, who accept offers only from firm 1; workers of the *low* type, who accept offers from both firms; and workers of the *null* type, who accept offers from no firm. It is straightforward to see that a worker with outside option r is of the null type iff $r \geq w_1$.

First, we show that if a worker, namely worker 1, with option r_1 accepts offers from both firms, then any other worker, namely worker 2, with outside option $r_2 < r_1$ also accepts offers from both firms. Suppose that worker 1 switches to the strategy of accepting offers from firm 1 only. Let $\Delta_{[t_f]}$ denote the *additional* amount of time that worker 1 will spend working at firm f after she switches. ($\Delta_{[t_f]} < 0$ means that worker 1 spends less time working at f.) Similarly, let $\Delta_{[t_\varnothing]}$ denote the additional amount of time that worker 1 is unemployed after she switches. Because workers choose actions optimally, we must have

$$r_1 \cdot \Delta_{[t_\varnothing]} + w_1 \cdot \Delta_{[t_\varnothing]} + w_2 \cdot \Delta_{[t_\varnothing]} < 0.$$

On the other hand, it is straightforward to observe that $\Delta_{[t_\varnothing]} > 0$, that is, worker 1 spends more time unemployed after her switch. Therefore, we should have

$$r_2 \cdot \Delta_{[t_\varnothing]} + w_1 \cdot \Delta_{[t_\varnothing]} + w_2 \cdot \Delta_{[t_\varnothing]} < 0.$$

This implies that worker 2 can increase her steady-state earnings if she accepts offers from both firms, which is a contradiction.

Now, let r^* be the infimum of r over all workers with outside option r who accept offers from firm 1. (Note that there exist such workers, because there is a positive mass of workers with an outside option greater than w_2.) This finishes the proof. $\square$

Definition F.3. *The cutoff representation of a nontrivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$ *is given by* $(\mathbf{P}, r, \mathbf{k})$*, where*

(i) $\mathbf{k}$ *denotes the customer composition given by* $\mathbf{A}$*, and*

(ii) r *denotes the cutoff obtained by applying Lemma F.2 on* Σ*.*

Observe that if the cutoff representations of two nontrivial subgame equilibria are equal, then those subgame equilibria are equal.

Definition F.4. *The extended cutoff representation of a nontrivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$ *is given by* $(\mathbf{P}, r, \theta, \mathbf{k})$*, where*

(i) $(\mathbf{P}, r, \mathbf{k})$ *is the cutoff representation of* Σ*, and*

(ii) $\theta = \frac{\lambda - mr}{k_{-f} + \lambda}$*, where* $f = \arg\max_{f \in \mathcal{F}} \{w_f\}$ *and* $\lambda = mw_f$*.*

Observe that $\theta = 0$ when $w_1 = w_2$. Variable θ has a simple interpretation: a fraction θ of the mass of idle workers accept offers only from the firm that offers the maximum wage. The value of θ (given in Definition F.4) is derived by solving the equation

$$mr = (\lambda - k)(1 - \theta) + k_f(1 - \theta) + k_{-f},$$

which computes the mass of workers who accept offers from both firms. The following fact uses the above equation to write r in terms of the rest of the parameters involved.

Fact F.5. *Let* $\Sigma = (\mathbf{P}, r, \theta, \mathbf{k})$ *be a nontrivial subgame equilibrium with* $w_1 \geq w_2$*. Then,*

$$r = \frac{(\lambda - k)(1 - \theta) + k_1(1 - \theta) + k_2}{m}.$$

Lemma F.6. *Let* $\Sigma = (\mathbf{P}, r, \theta, \mathbf{k})$ *be a nontrivial subgame equilibrium with* $w_1 \geq w_2$*. Then, we must have* $(1 - \theta)(mw_1 - k) = mw_2 - k$*.*

Proof. The proof is trivial when $w_1 = w_2$. Without loss of generality, suppose $w_1 > w_2$. Observe that r, θ are related by

$$r = \frac{(\lambda - k)(1-\theta) + k_1(1-\theta) + k_2}{m}, \tag{F.1}$$

which holds because exactly a fraction $1-\theta$ of the busy workers at firm 1 and all the busy workers at firm 2 must be accepting offers from both firms.

Define $c_2 = c(i \cdot (1-\theta))$. Now, observe that

$$\theta = 1 - \frac{c^{-1}(c_2)}{mw_1 - k}, \tag{F.2}$$

where we recall that $i = \lambda - k$, by definition.

Next, we plug in (F.1) and (F.2) into the worker's indifference condition and solve the equation for w_1. This lets us write w_1 as follows:

$$w_1 = \frac{c^{-1}(c_2) \cdot (k_2 - mw_1) + k_1 k_2 + (k_2 - mw_1)(k_2 - mw_2)}{k_1 m}$$

Rearranging the terms of the above equality implies

$$0 = (k_2 - mw_1)(c^{-1}(c_2) + k_1 + k_2 - mw_2).$$

Now, note that $k_2 < mw_2 \leq mw_1$ must always hold, and therefore, according to the above equality, we must have

$$c^{-1}(c_2) = mw_2 - k,$$

which implies $c_2 = c(mw_2 - k)$. □

Proposition F.7. *In any nontrivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, the waiting cost incurred by customers at firm f is $c(mw_f - k)$.*

Proof. The proof is a direct consequence of Lemma F.6. □

Definition F.8. *A payment profile $((p_1, w_1), (p_2, w_2))$ is called standard if it induces a non-trivial subgame equilibrium Σ such that $c'(mw_1 - k) = c'(mw_2 - k) = \frac{-1}{m}$, where k is the total rate of customers requesting service in Σ.*

F.3 Missing proofs from Section 6.1

Lemma F.9. *Consider the following continuous-time stochastic process with state space $V = \{0, 1, 2\}$. The transition rate from state 0 to i is λ_i, for $i \in \{1, 2\}$. After a transition from state 0 to state i, the process remains at state i for a unit of time, after which it returns to state 0. Let π_i denote the fraction of time that the process spends at state i.*[14] *Then, $\pi_i = \frac{\lambda_i}{1+\lambda_1+\lambda_2}$ for $i > 0$ and $\pi_0 = \frac{1}{1+\lambda_1+\lambda_2}$.*

Proof. A straightforward Law of Large Numbers argument implies that $\pi_1 = \pi_2 \cdot \frac{\lambda_1}{\lambda_2}$ and $\pi_0(\lambda_1 + \lambda_2) = \pi_1 + \pi_2$. These two equations together with $\sum_{i=0}^{2} \pi_i = 1$ prove the claim. $\square$

Fact F.10. *Given a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, there is no worker for whom there are only two actions $S = \{1\}, T = \{2\}$ such that both provide her the maximum steady-state earnings.*

Proof. Proof by contradiction. Note that the worker's payoff under either action should be positive, because otherwise the action $\varnothing$ attains the same level of steady-state earnings, which is a contradiction. Now, suppose that $w_2 \leq w_1$ without loss of generality. It is straightforward to see that action $\{1, 2\}$ provides a larger level of steady-state earnings than action $\{2\}$, which is a contradiction. $\square$

Proof of Proposition 6.2. The proof is by contradiction. Suppose there exist two nontrivial subgame equilibria, namely $\Sigma = (\mathbf{P}, \mathbf{A})$, and $\Sigma' = (\mathbf{P}, \mathbf{A}')$. Let $\mathbf{k}$ and $\mathbf{k}'$ respectively denote the customer compositions in $\mathbf{A}$ and $\mathbf{A}'$. Suppose $\mathbf{k} = (k_1, k_2)$ and $\mathbf{k}' = (k'_1, k'_2)$, and let $k = k_1 + k_2$ and $k' = k'_1 + k'_2$.

By Lemma F.6, the waiting cost that customers incur at firm f in Σ equals $c(mw_f - k)$. Let $b_f = p_f + c(mw_f - k)$ denote the aggregate cost that customers incur at firm f in Σ.[15] Define b'_f similarly for Σ'. Furthermore, with a slight abuse of notation, we define the function $b_f : \mathbb{R} \to \mathbb{R}$ as

$$b_f(\hat{k}) = p_f + c(mw_f - \hat{k}),$$

where we have taken $\hat{k}$ to be a variable. Using this notation, we write the market clearing condition, which states that

$$\hat{k} = D(b_1(\hat{k}), b_2(\hat{k})) + D(b_2(\hat{k}), b_1(\hat{k})).$$

[14] To state this more precisely, one should define $\pi_{i,t}$ to be the fraction of time that the process spends at state i from the start of the process until time t, and then define $\pi_i = \lim_{t\to\infty} \pi_{i,t}$.

[15] When we defined the notion of worker composition, we noted that $b(f)$ denotes the mass of workers busy at firm f. This notation is not used in this proof and is irrelevant to b_f.

(Recall the definition of the demand function D from Section F.1.) In other words, the condition says that the mass of busy drivers must equal the rate of customers who are served. Observe that the LHS is strictly increasing in $\hat{k}$, whereas the RHS is decreasing in $\hat{k}$. Therefore, this equation has a unique solution, which we denote by k^*. Note that we must have $k = k^*$ and $k' = k^*$. Therefore, $k = k'$. But then this implies that $b_f = b'_f$ for all f (because $b'_f = p'_f + c(mw'_f - k')$, by definition). Therefore,

$$k_f = D(b_f(k), b_{-f}(k)) = k'_f$$

must hold for all f, which implies that $\mathbf{k} = \mathbf{k}'$. This, together with Lemma F.6, implies that the cutoff representations of Σ, Σ' are identical, which means that Σ, Σ' are identical. □

Lemma F.11. *For any payment profile* $\mathbf{P}$ *and any firm* $f \in \mathcal{F}$*, there exists at most one trivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$ *satisfying* $k_f > 0$*.*

Proof. Suppose Σ is such a trivial subgame equilibrium, and let $\mathbf{k}$ denote the customer composition in $\mathbf{A}$. In Σ, all workers with an outside option less than w accept offers from firm f, and all other workers reject them. Therefore, any trivial subgame equilibrium in which f serves a nonzero rate of customers has the same worker composition as Σ.

On the other hand, we see that the waiting cost that customers incur at firm f in Σ is equal to $c(mw_f - k)$. Recall the market-clearing condition (3.1), according to which we can write

$$k = 1 - p_f - c(mw_f - k)$$

for Σ. Observe that the LHS is strictly increasing in k, whereas the RHS is decreasing. Therefore, this equation has a unique root, k. This implies that any trivial subgame equilibrium in which f serves a non-zero rate of customers must have the same customer composition as Σ. This completes the proof. □

F.3.1 The selection rule: proving uniqueness

Recall that, in Section 6.1, we defined $\Sigma_{\mathbf{P}}$ as the unique steady-state subgame equilibrium that serves the highest rate of customers. We will prove the promised uniqueness in Lemma F.12. There, we will show that $\Sigma_{\mathbf{P}}$ is the unique trivial subgame equilibrium under $\mathbf{P}$ in which the rate of customers served is the highest, if a nontrivial subgame equilibrium exists under $\mathbf{P}$. Otherwise, $\Sigma_{\mathbf{P}}$ is the unique nontrivial subgame equilibrium in which the rate of customers served is the highest, if no nontrivial subgame equilibrium exists under $\mathbf{P}$.

We first need a few definitions. By Proposition F.7, the waiting cost that customers at firm f incur in $\Sigma = (\mathbf{P}, \mathbf{A})$ equals $c(mw_f - k)$. Let $b_f = p_f + c(mw_f - k)$. Furthermore, with a slight abuse of notation, we define the function $b_f : \mathbb{R} \to \mathbb{R}$ as

$$b_f(\hat{k}) = p_f + c(mw_f - \hat{k}), \tag{F.3}$$

where we have taken $\hat{k}$ to be a variable. Using this notation, we write the market clearing condition

$$\hat{k} = D(b_1(\hat{k}), b_2(\hat{k})) + D(b_2(\hat{k}), b_1(\hat{k})). \tag{F.4}$$

Observe that the LHS is strictly increasing in $\hat{k}$, whereas the RHS is decreasing in $\hat{k}$. Further more, the RHS is at most 1 at $\hat{k} = 0$, and is 0 at $\hat{k} = 1$. So, the above equation has a unique solution, which we denote by k^*.

Given k^*, we define $\Sigma_{\mathbf{P}} = (\mathbf{P}, \mathbf{A})$ as follows:

1. The customer composition $\mathbf{k}$ satisfies $k_1 = D(b_1(k^*), b_2(k^*))$ and $k_2 = D(b_2(k^*), b_1(k^*))$. Furthermore, $k = k^*$, where we recall that $k \equiv k_1 + k_2$ by definition.

2. If $k_f = 0$ for some firm f, then a nontrivial subgame equilibrium does not exist (see Lemma F.12 below.) The selection rule chooses the trivial subgame equilibrium in which workers do not accept offers from firm f. Workers with an outside option r accept offers of $-f$ iff $k_{-f} > 0$ and $r < w_{-f}$.

3. If $k_1, k_2 > 0$, then a nontrivial subgame equilibrium exists. Let $(\mathbf{P}, r^*, \theta, \mathbf{k})$ denote the extended cutoff representation of $\Sigma_{\mathbf{P}}$. Workers with outside option $r \in [0, r^*]$ accept offers from both firms. Workers with outside option $r \in (r^*, \max\{w_1, w_2\}]$ accept offers from firm 1 only. The rest of the workers accept offers from neither firm.

Lemma F.12. *The subgame equilibrium defined above, $\Sigma_{\mathbf{P}} = (\mathbf{P}, \mathbf{A})$, is the unique subgame equilibrium that serves the highest rate of customers among all possible subgame equilibria induced by $\mathbf{P}$. Furthermore, if there exists a nontrivial subgame equilibrium under $\mathbf{P}$, then $\Sigma_{\mathbf{P}}$ is that equilibrium.*

Proof. First, we prove the second part. Suppose a nontrivial subgame equilibrium exists, namely $\Sigma' = (\mathbf{P}, \mathbf{A}')$. We will show that $\Sigma' = \Sigma_{\mathbf{P}}$. To this end, we plug $\hat{k} = k'$ into (F.3).

This gives

$$b_f(\hat{k}) = b'_f, \quad \forall f \in \mathcal{F},$$

where $b'_f = p_f + c(mw_f - k')$. Therefore, we must also have

$$D(b_1(\hat{k}), b_2(\hat{k})) = k'_1,$$
$$D(b_2(\hat{k}), b_1(\hat{k})) = k'_2.$$

The above equations imply that (F.4) is satisfied when $\hat{k} = k'$. Proposition 6.2 and the definition of k^* together imply that $\Sigma_{\mathbf{P}} = \Sigma'$.

It remains to prove the first part of the lemma. The proof is done for two separate cases: either $\Sigma_{\mathbf{P}}$ is a nontrivial subgame equilibrium or it is not.

Case 1. Suppose that $\Sigma_{\mathbf{P}}$ is a nontrivial subgame equilibrium. The proof is by contradiction. Consider another subgame equilibrium, namely $\Sigma' = (\mathbf{P}, \mathbf{A}')$, for which $k' \geq k$. By Proposition 6.2, Σ' must be a trivial subgame equilibrium. Without loss of generality, suppose that $k'_2 = 0$ in Σ'. This also implies that $k'_1 = k'$. Observe that $b'_1 = p_1 + c(mw_1 - k')$. Therefore, we must have

$$b'_1 = p_1 + c(mw_1 - k') \geq p_1 + c(mw_1 - k) = b_1.$$

But $b'_1 \geq b_1$ implies that $k' < k$, which is a contradiction.

Case 2. Suppose that $\Sigma_{\mathbf{P}}$ is a trivial subgame equilibrium. Without loss of generality, suppose that $k_2 = 0$. There is at most one other subgame equilibrium, namely $\Sigma' = (\mathbf{P}, \mathbf{A}')$, for which $k'_1 = 0$. For the sake of contradiction, suppose that $\Sigma \neq \Sigma'$ and $k \leq k'$. Without loss of generality, suppose that $k'_1 = 0$ in Σ'. Observe that $k'_2 = k'$ and $k_1 = k$. By the definition of the ascending process that finds $\Sigma_{\mathbf{P}}$, we must have

$$b_2 = p_2 + c((mw_2 - k)_+) \geq 1, \tag{F.5}$$

where the notation $(x)_+$ denotes the positive part of x. On the other hand, note that

$$b'_2 = p_2 + c(mw_2 - k') \geq p_2 + c((mw_2 - k)_+) = b_2 \geq 1,$$

where the last inequality holds by (F.5). We just showed that $b_2' \geq 1$. Therefore, we must have $k' = k_2' = 0$. Consequently, $\Sigma = \Sigma'$, which is a contradiction. $\square$

Fact F.13. *Both firms serve the same rate of customers at any symmetric duopoly equilibrium.*

Proof. First, we show that Σ is nontrivial unless both firms serve 0 customers. For contradiction, suppose that firm 1 serves a positive rate of customers whereas firm 2 serves 0 customers. Observe that firm 2 gains a positive profit if it increases the wage slightly, which results in a contradiction. Therefore, for the rest of the proof we can assume that Σ is nontrivial .

Let k_f denote the steady-state rate of customers who join firm f in Σ, and let $k = k_1 + k_2$. Also, let p, w respectively denote the price and wage at both firms. Define $b_f = p + c(mw - k)$, and observe that $b_1 = b_2$. Define $b = b_1$. Now, observe that $k_1 = D(b, b)$ and $k_2 = D(b, b)$ (which holds because all workers accept offers from both firms in Σ). $\square$

G Proof of Theorem 6.5: profit maximizing firm

This section presents the proof of Theorem 6.5 for the case of a profit-maximizing firm. The proof involves several steps. We go over a brief proof sketch before presenting the formal proof. We start by stating a system of necessary conditions that any duopoly equilibrium must satisfy (Section G.1). In fact, these conditions are necessary and sufficient for a *local* duopoly equilibrium , i.e., a duopoly equilibrium in which no firm has an incentive to make infinitesimal deviations from its payment profile to a "nearby" payment profile. We then show that these conditions have a unique solution in the interval $(\underline{m}, \hat{m})$. The system of equations that characterize the local duopoly equilibrium has a closed-form solution. We use the closed-form solution to prove the claim of the theorem for the local duopoly equilibrium (Section G.2). In the last step of the proof, we show that this unique solution in fact represents a (global) duopoly equilibrium (Section G.3).

G.1 Local duopoly equilibrium

Definition G.1. A nontrivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is called a *local* duopoly equilibrium if there exists an open ball $B \subset \mathbb{R}^2 \times \mathbb{R}^2$ around $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ such that for any firm f and any $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ with $\overline{\mathbf{P}} \in B$, we have $\Pi_f(\Sigma_{\mathbf{P}}) \geq \Pi_f(\Sigma_{\overline{\mathbf{P}}})$.

Fact G.2. In Definition G.1, when given a nontrivial subgame equilibrium Σ, one can always choose the ball B such that any $\overline{\mathbf{P}} \in B$ induces a nontrivial subgame equilibrium, which will also be unique by Proposition 6.2. Therefore, $\Sigma_{\overline{\mathbf{P}}}$ would be the unique nontrivial subgame equilibrium under $\overline{\mathbf{P}}$.

We use Fact G.2 to write the conditions that characterize a local duopoly equilibrium. First, we need to define a notation.

Definition G.3. *Let $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ be defined as follows: for any payment profile $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$, $D(\mathbf{P}_f, \mathbf{P}_{-f})$ is the steady-state rate of customers who join firm f in $\Sigma_{\mathbf{P}}$.*[16]

For notational simplicity, we sometimes use $D(p_1, w_1; p_2, w_2)$ to denote $D(\mathbf{P}_1, \mathbf{P}_2)$, where $\mathbf{P}_1 = (p_1, w_1)$ and $\mathbf{P}_2 = (p_2, w_2)$. Also, we use the notations $D_1(p_1, w_1; p_2, w_2)$ and $D_2(p_1, w_1; p_2, w_2)$ respectively to denote the derivatives of D with respect to its first and second arguments (i.e., price p_1 and wage w_1).

We are now ready to state the conditions that characterize a local duopoly equilibrium.

Proposition G.4. *A subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is a local duopoly equilibrium iff it satisfies*

$$k_1 = -D_1(p_1, w_1; p_2, w_2) \cdot (p_1 - w_1), \tag{G.1}$$

$$k_1 = D_2(p_1, w_1; p_2, w_2) \cdot (p_1 - w_1), \tag{G.2}$$

$$k_1 = D(p_1, w_1; p_2, w_2), \tag{G.3}$$

$$k_2 = -D_1(p_2, w_2; p_1, w_1) \cdot (p_2 - w_2), \tag{G.4}$$

$$k_2 = D_2(p_2, w_2; p_1, w_1) \cdot (p_2 - w_2), \tag{G.5}$$

$$k_2 = D(p_2, w_2; p_1, w_1). \tag{G.6}$$

Proof. First, we show that any local duopoly equilibrium must satisfy the given conditions. (G.1) and (G.2) are the first-order conditions of firm 1 for price and wage, respectively. (G.3) is a balance equation: on the left-hand side we have the rate of customers who join the firm, and on the right-hand side we have the rate of customers who gain positive payoff from joining the firm. This could also be interpreted as a market-clearing condition. Equations (G.4),(G.5), and (G.6) are the same equations but written for firm 2.

On the other hand, if the given equations are satisfied for a given Σ, then it is a local duopoly equilibrium. To see why, observe that the first-order conditions for firm f imply

[16] Note that we use the symmetry of customer demand for the two firms in writing such a functional form for determining the rate of customers who join a firm.

that $\nabla\Pi_f(p_f, w_f) = 0$. This guarantees the existence of a ball B around $\mathbf{P}$ that satisfies the condition given in Definition G.1. $\square$

Corollary G.5 (of Proposition G.4)**.** *If the tuples (p_1, w_1, k_1) and (p_2, w_2, k_2) with $k_1, k_2 > 0$ satisfy the conditions given in Proposition G.4, then the payment profile $\mathbf{P} = ((p_1, w_1), (p_2, w_2))$ induces a unique nontrivial subgame equilibrium Σ that is also a local duopoly equilibrium.*

Proof. The proof is similar to the proof of the second part of Proposition G.4; in addition to that, observe that conditions (G.3) and (G.6) guarantee that the payment profile $\mathbf{P}$ induces a nontrivial subgame equilibrium. The uniqueness is implied by Proposition 6.2. $\square$

Proposition G.6. *A subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ with $\mathbf{P}_f = (p, w)$ for all $f \in \mathcal{F}$ and $\mathbf{k} = (k_1, k_2) = (k/2, k/2)$ is a symmetric local duopoly equilibrium iff it satisfies*

$$k/2 = -D_1(p, w; p, w) \cdot (p - w), \tag{G.7}$$

$$c'(mw - k) = \frac{-1}{m}, \tag{G.8}$$

$$k/2 = D(p, w; p, w). \tag{G.9}$$

Proof. First, we show that a local duopoly equilibrium satisfies the given conditions. (G.7) is the first-order condition for price, which is identical for both firms by symmetry. (G.9) is the balance equation, which is identical for both firms. On its left-hand side we have the rate of customers who join the firm, and on its right-hand side we have the rate of customers who gain positive payoff from joining the firm. This condition could also be interpreted as a market-clearing condition. To derive (G.8), observe that if (G.8) does not hold, then the firm could increase its profit by changing price and wage: if $c'(mw_1 - k) > \frac{-1}{m}$, then firm 1 could increase its profit by decreasing wage by some sufficiently small $\epsilon > 0$ and decreasing price by a positive $\epsilon' < \epsilon$, while doing this so that the aggregate cost offered to its customers does not change. Hence, the rate of the customers that the firm serves would not change, but the firm's commission fee would go up, and therefore the firm's profit. If $c'(mw_1 - k) < \frac{-1}{m}$, then the same could be done, but by increasing wage and price. Condition (G.8) could also be derived in another way, similar to what we showed in Proposition A.3 for the case of a monopoly. We also include the second way of deriving Condition (G.8): the first-order condition of a firm with respect to wage is

$$k/2 = D_2(p, w; p, w) \cdot (p - w). \tag{G.10}$$

Equating the right-hand side of the above equation with the right-hand side of (G.7) implies (G.8). This is shown in files "duo-focp" and "duo-focw", which compute the first-order conditions with respect to price and wage. The right-hand side of the first-order conditions should be equal (because their left-hand sides are equal); equating the right-hand sides implies (G.8).

Next, we show that if a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ satisfies the above conditions, then it should be a local duopoly equilibrium. To this end, we prove that (G.7) and (G.8) together imply that the first-order condition of a firm f with respect to wage holds. This is derived from the same exercise we just performed: we saw that equating the right-hand sides of (G.7) and (G.10) implies (G.8). Similarly, we show that (G.7) and (G.8) together imply (G.10). This is shown in file "retrieving-focw". □

Corollary G.7 (of Proposition G.6)**.** *If the tuple (p, w, k) with $k > 0$ satisfies the conditions given in Proposition G.6, then the payment profile $\mathbf{P} = ((p, w), (p, w))$ induces a unique nontrivial subgame equilibrium Σ which is also a local duopoly equilibrium.*

Proof. The proof is similar to the proof of the second part of Proposition G.6; in addition to that, observe that (G.9) guarantees that the payment profile $\mathbf{P}$ induces a nontrivial subgame equilibrium. The uniqueness is implied by Proposition 6.2. □

We also need to define some additional notation for the notion of monopoly equilibrium. We define the monopoly equilibrium in this setting by excluding firm 2 and allowing firm 1 to choose its optimal price and wage. In this sense, the definition of the monopoly equilibrium remains the same as in Section 3. To help readability, we repeat the conditions that characterize the monopoly equilibrium in terms of the demand function D. We let $D(p, w; \infty, 0)$ denote the rate of customers who join firm 1 when its payment profile is (p, w) and firm 2 is excluded. Note that excluding firm 2 is equivalent to setting its payment profile to $(\infty, 0)$.

Proposition G.8. *A tuple (p, w, k) defines a monopoly equilibrium under firm 1, with p, w being the price and wage at firm 1 and k being the rate of customers served by firm 1, iff the tuple satisfies the following conditions:*

$$k = -D_1(p, w; \infty, 0) \cdot (p - w), \tag{G.11}$$

$$c'(mw - k) = \frac{-1}{m}, \tag{G.12}$$

$$k = D(p, w; \infty, 0). \tag{G.13}$$

Proof. The proof follows from the analysis of the monopoly equilibrium in Section 4. □

Definition G.9. *Given m, let $\mathsf{DE}(m)$ denote the system of three equations given in Proposition G.6, and let $\mathsf{ME}(m)$ denote the system of three equations given in Proposition G.8.*

Lemma G.10. *Let $m_0 = \frac{1}{-c'(0)}$. When $m \leq m_0$, there is no tuple (p, w, k) with $k > 0$ that satisfies $\mathsf{DE}(m)$ or $\mathsf{ME}(m)$.*

Proof. The proof is based on equations (G.8) and (G.12). First, suppose that $m \leq m_0$ and there exists a tuple (p, w, k) with $k > 0$ that satisfies $\mathsf{DE}(m)$. Define $i = mw - k$. Note that $i \geq 0$ must hold; otherwise the (p, w, k) is not a valid solution, since the argument of c falls outside of its domain. Furthermore, $i > 0$ should hold; otherwise $k > 0$ cannot hold because c is a regular cost function.

Because c is convex, its derivative is increasing. Therefore, $c'(i) > c'(0)$ must always hold, which implies that $\frac{-1}{m} > c'(0)$. Consequently, $m > -\frac{1}{c'(0)}$ holds, which proves the claim for the system $\mathsf{DE}(m)$. The proof for the system $\mathsf{ME}(m)$ is the same. □

G.2 Proving the theorem's claim for local equilibria

We recall Lemma G.10 and set $\underline{m} = m_0$. In this step of the proof, we show the existence of $\hat{m} > m_0$ such that for any $m \in (m_0, \hat{m})$, the following holds: (i) there is a unique solution to $\mathsf{DE}(m)$, which we denote by the tuple $(p_{\mathsf{duo}}(m), w_{\mathsf{duo}}(m), k_{\mathsf{duo}}(m))$, (ii) there is a unique solution to $\mathsf{ME}(m)$, which we denote by the tuple $(p_{\mathsf{mon}}(m), w_{\mathsf{mon}}(m), k_{\mathsf{mon}}(m))$, and (iii) $p_{\mathsf{duo}}(m) > p_{\mathsf{mon}}(m)$, $u^C_{\mathsf{duo}}(m) < u^C_{\mathsf{mon}}(m)$, $w_{\mathsf{duo}}(m) > w_{\mathsf{mon}}(m)$, and $u^W_{\mathsf{duo}}(m) > u^W_{\mathsf{mon}}(m)$.

The value for $\hat{m}$ will be set by the end of the proof.

Lemma G.11. *There exists m' such that for all $m \in (\underline{m}, m')$, there exists a unique monopoly equilibrium at m.*

Proof. First, observe that for any $m > \underline{m}$, there exists at least one monopoly equilibrium at m. This is implied by Lemma B.1. To prove the existence of m' and the uniqueness of equilibrium in the interval $(\underline{m}, m')$, we find the closed-form expressions for all the (possibly complex) roots of $\mathsf{ME}(m)$. This is done in file "ME-sols". This system reduces to an equation of degree 3 in p, and therefore, there are 3 (possibly complex) solutions to the system. We retrieve the three solutions. We use $k^1(m), k^2(m), k^3(m)$ to denote the values that each of these solutions assigns to the variable k, as a function of m. We observe that $k^1(m) \equiv 0$, and $\lim_{m \to m_0} k^2(m) > 0$; this rules out two of the solutions in an interval $(\underline{m}, m')$, when m' is sufficiently close to m. Therefore, there exists at most one monopoly equilibrium in the interval $(\underline{m}, m')$. On the other hand, recall from Lemma B.1 that any $m > \underline{m}$ is feasible, which means that there exists at least one monopoly equilibrium when $m > \underline{m}$. This completes the proof. $\square$

Lemma G.12. *There exists m' such that for all $m \in (\underline{m}, m')$, there exists a unique symmetric local duopoly equilibrium at m.*

Proof. First, we need to recall a notation from Section F.1: $D : [0,1]^2 \to [0,1]$ is the demand function of customers in terms of the aggregate costs that the firms offer, i.e., $D(b_1, b_2)$ determines the rate of customers demanding to join firm 1 assuming that the aggregate cost at every firm f is b_f.

First, we show that m' can be set so that for any $m \in (\underline{m}, m')$, there exists at least one local duopoly equilibrium at m. (While this claim holds for all $m > \underline{m}$, we only prove it for this special case to make the proof simpler.) After that, we prove the uniqueness of the local duopoly equilibrium for all $m \in (\underline{m}, m')$. We will set the value of m' during the proof.

Fix m, and consider a continuum of subgame equilibria, $\{\Sigma^\beta : \beta \in [0, \bar{\beta}]\}$, defined as follows: $\bar{\beta} < 1$ is a positive constant which we will define later together with m'. Σ^β is a subgame equilibrium defined as follows. The variable $b^\beta \equiv 1 - \beta$ represents the aggregate cost that is offered to customers in Σ^β. Therefore, Σ^β is just the $\varnothing$ subgame equilibrium for $\beta = 0$. For positive β, $\Sigma^\beta = (\mathbf{P}^\beta, \mathbf{A}^\beta)$ is defined as the symmetric subgame equilibrium with customer composition

$$\mathbf{k}^\beta = (k_1^\beta, k_2^\beta) = (D(b^\beta, b^\beta), D(b^\beta, b^\beta))$$

and payment profile $\mathbf{P}^\beta = ((p^\beta, w^\beta), (p^\beta, w^\beta))$ such that

$$p^\beta + c(mw^\beta - k^\beta) = b^\beta, \tag{G.14}$$

$$c'(mw^\beta - k^\beta) = \frac{-1}{m}, \tag{G.15}$$

where $k^\beta = k_1^\beta + k_2^\beta$. It is straightforward to see that given any fixed b^β, there is a unique pair (p^β, w^β) that satisfies (G.14) and (G.15): the value of b^β uniquely determines k^β, which uniquely determines w^β by (G.15). This in turn uniquely determines p^β, by (G.14).

Define $\mathfrak{L}(\beta)$ as

$$\mathfrak{L}(\beta) = \left(-\frac{D_1(p^\beta, w^\beta; p^\beta, w^\beta)}{D(p^\beta, w^\beta; p^\beta, w^\beta)} \right) \cdot (p^\beta - w^\beta). \tag{G.16}$$

The significance of this definition is that when Σ^β is a local duopoly equilibrium, we must have $\mathfrak{L}(\beta) = 1$; only then will (G.16) coincide with the firm's first-order condition for price, as given in (G.1).

In the rest of the proof, we will show that (i) $\lim_{\beta\to 0}\mathfrak{L}(\beta)=\infty$, (ii) $\lim_{\beta\to\bar{\beta}}\mathfrak{L}(\beta)=0$, and (iii) $\mathfrak{L}(\beta)$ is continuous at any $\beta\in(0,\bar{\beta})$. Proving these three steps completes the proof: They imply that $\mathfrak{L}(\beta^*)=1$ for some $\beta^*\in(0,\bar{\beta})$. It is then straightforward to verify that Σ^{β^*} satisfies all the conditions in Proposition G.6, and therefore it is a symmetric local duopoly equilibrium.

Proof for step (i). There are two multiplicands on the right-hand side of (G.16). We denote the first one by $f(\beta)$ and the second one by $g(\beta)$. We will prove that $\lim_{\beta\to 0} f(\beta)=\infty$, and $\lim_{\beta\to 0} g(\beta)>0$. This would complete this step.

In file "f(beta)", we derive the closed-form expression for $f(\beta)$:

$$f(\beta)=\frac{2\left(a^2m-am+b^\beta-1\right)}{(1-b^\beta)\left(a^2m-am+2b^\beta-2\right)}, \tag{G.17}$$

where $a=1-\sigma$. This implies that $\lim_{\beta\to 0}\mathfrak{L}(\beta)=\infty$. To finish this step, it remains to prove $\lim_{\beta\to 0} g(\beta)>0$.

Claim G.13. *There exists a nontrivial subgame equilibrium $\Sigma^*=(\mathbf{P}^*,\mathbf{A}^*)$ with a customer composition $\mathbf{k}^*=(k_1^*,k_2^*)$ such that $c'(mw_1^*-k^*)=\frac{-1}{m}$.*

Proof. Because $m>\underline{m}=\frac{1}{-c'(0)}$, then there exists some nontrivial subgame equilibrium $\Sigma'=(\mathbf{P}',\mathbf{A}')$ at m. The proof for this fact is essentially identical to the proof of Lemma B.1. We will show that Σ' can be chosen such that it satisfies the condition in the claim statement. Let $\mathbf{k}'$ denote the customer composition in Σ'. Among all nontrivial subgame equilibria which have the same customer composition as $\mathbf{k}'$, let $\Sigma^*=(\mathbf{P}^*,\mathbf{A}^*)$ be the one in which firm 1 earns the highest profit. We will show that, then, $c'(mw_1^*-k^*)=\frac{-1}{m}$ must hold. The proof is by contradiction; suppose this is not the case.

First, recall the expression that Proposition F.7 provides for the waiting cost of customers in any nontrivial subgame equilibrium: the waiting cost at firm f in Σ^* is equal to $c(mw_f^*-k^*)$. First, suppose that $c'(mw_1^*-k^*)>\frac{-1}{m}$. Firm 1 then can decrease its wage by a sufficiently small $\epsilon>0$ and decrease its price by a positive $\epsilon'<\epsilon$ such that the aggregate cost offered to its customers does not change. Hence, the rate of the customers that firm 1 serves would not change, but the firm's commission fee (the difference between price and wage) would go up, and therefore, the firm's profit. This contradicts the definition of $\mathbf{P}^*$.

The proof for the case of $c'(mw_1^*-k^*)<\frac{-1}{m}$, analogous, but by increasing wage and price, instead of decreasing them as above. □

Let b_f^* denote the aggregate cost of firm f in Σ^*. Observe that for any β, customers incur the same waiting cost in Σ^β and in Σ^*, because $c'(mw_1^* - k^*) = c'(mw_1^\beta - k^\beta) = \frac{-1}{m}$. However, note that $b_1^* < b_1^\beta$ holds for β sufficiently close to 0. The two latter facts together imply that that $p_1^* < p^\beta$ holds for β sufficiently close to 0.

Also observe that $w_1^* > w^\beta$ holds for β sufficiently close to 0: because $c'(mw_1^* - k^*) = c'(mw_1^\beta - k^\beta) = \frac{-1}{m}$, and because $k^\beta < k^*$ for β sufficiently close to 0.

We have shown that $p_1^* < p^\beta$ and $w_1^* > w^\beta$ hold for β sufficiently close to 0. Therefore, we must have

$$\lim_{\beta \to 0} p^\beta - w^\beta > p_1^* - w_1^*.$$

Consequently, $\lim_{\beta \to 0} g(\beta) > 0$, and this step is complete.

Proof for step (ii). Consider the family of all nontrivial subgame equilibria with payment profiles $((p, w), (p, w))$ such that $p = w$. Let $\underline{b}$ denote the infimum of the aggregate costs that customers incur in this family. Define $\overline{\beta} = 1 - \underline{b}$. It is straightforward to verify that $\lim_{\beta \to \overline{\beta}} g(\beta) = 0$, i.e., as β approaches $\overline{\beta}$, the commission fee in Σ^β approaches 0.

To complete this step, it remains to show that $\lim_{\beta \to \overline{\beta}} f(\beta)$ exists and is finite. Recall from (G.17) that

$$f(\beta) = \frac{2\left(a^2 m - am + b^\beta - 1\right)}{\left(1 - b^\beta\right)\left(a^2 m - am + 2b^\beta - 2\right)},$$

where $a = 1 - \sigma$. We choose m' sufficiently close to $\underline{m}$ such that $\overline{\beta} < \frac{1-\sigma}{2}$. This guarantees that the right-hand side of the above equation is finite for all $\beta \leq \overline{\beta}$, and therefore ensures the continuity of $f(\beta)$ for all such β.[17] Hence, $\lim_{\beta \to \overline{\beta}} f(\beta)$ exists and is finite. This completes step (ii).

Proof for step (iii). To complete this step, we need to show that $f(\beta)$ and $g(\beta)$ are continuous in the interval $(0, \bar{\beta})$. We proved the claim for $f(\beta)$ in step (ii). It remains to prove the claim for $g(\beta)$. Recall that $g(\beta) = p^\beta - w^\beta$, and observe that, by (G.15),

$$p^\beta = \beta - c(c'^{-1}(-1/m)),$$
$$w^\beta = \frac{c'^{-1}(-1/m) + D(b^\beta, b^\beta)}{m},$$

[17] We note that the continuity holds for larger m', but we make the stronger assumption to keep the proof simple.

which are continuous functions in β. This proves that $g(\beta)$ is continuous in the interval $(0, \bar{\beta})$, and completes step (iii).

So far we have shown the existence of at least one symmetric local duopoly equilibrium. To complete the proof of the lemma, we need to show the uniqueness of the equilibrium: We will show that m' can be chosen such that for all $m \in (\underline{m}, m')$, there exists a unique symmetric local duopoly equilibrium at m. To this end, we find the closed-form expressions for all the (possibly complex) roots of $\mathsf{DE}(m)$. This is done in file "DE-sols". This system reduces to an equation of degree 4 in p, and therefore, there are 4 (possibly complex) solutions to the system. We retrieve the 4 solutions. We use $k^1(m), k^2(m), k^3(m), k^4(m)$ to denote the values that each of these solutions assign to the variable k, as a function of m. We observe that $k^1(m) \equiv 0$, and that $\lim_{m \to m_0} k^2(m) > 0$ and $\lim_{m \to m_0} k^4(m) > 0$. This rules out 3 of the solutions in an interval $(\underline{m}, m')$. Therefore, there exists at most one symmetric local duopoly equilibrium in the interval $(\underline{m}, m')$. This completes the proof of the lemma. $\square$

Lemma G.14. *There exists m' such that for all $m \in (\underline{m}, m')$, a unique monopoly equilibrium and a unique symmetric duopoly equilibrium exist at m, and furthermore, $p_{\mathsf{duo}}(m) > p_{\mathsf{mon}}(m)$ and $u^C_{\mathsf{duo}}(m) < u^C_{\mathsf{mon}}(m)$, whereas $w_{\mathsf{duo}}(m) > w_{\mathsf{mon}}(m)$ and $u^W_{\mathsf{duo}}(m) > u^W_{\mathsf{mon}}(m)$.*

The existence and uniqueness of the equilibria in the above lemma are implied by Lemma G.11 and Lemma G.12. It remains to prove the second part of the claim. This will be done in Sections G.2.1, G.2.2, G.2.3, and G.2.4.

G.2.1 Proof of Lemma G.14: Preliminaries

Consistent with our usual notation, we use the variable b to denote $p + c(mw - k)$. For example, $b_{\mathsf{mon}} = p_{\mathsf{mon}} + c(mw_{\mathsf{mon}} - k_{\mathsf{mon}})$. This quantity is also called the *aggregate cost* that customers incur. In the case of monopoly, we let the function $D : \mathbb{R}_+ \to [0, 1]$ determine the customers' demand function, i.e., $D(b)$ is the rate of customers who request service when the aggregate cost that a customer incurs at the firm is b. Similarly, and with slight abuse of notation, we use $D(b_1, b_2)$ in the case of duopoly to denote the rate of customers who join firm 1 when the aggregate costs at firms 1,2 respectively are b_1, b_2. Recall that this function is defined in Section F.1.

For the case of monopoly, the *adjusted price f(beta)* of customers' demand when the firm's price is p and its wage is fixed at w is defined by

$$\mathfrak{E}_{\mathsf{mon}}(p; w) = -\frac{k(p; w)'(p)}{k(p; w)} \cdot (p - w) = -\frac{D'(b(p; w)) \cdot b'(p; w)}{D(b(p; w))} \cdot (p - w), \tag{G.18}$$

where the parameters in the above equation are defined as follows. The function $k(p;w)$ denotes the rate of customers who join the firm as a function of the price posted by the firm, p, while holding wage fixed at w. Similarly, $b(p;w)$ denotes the aggregate cost that customers face as a function of p, while holding wage fixed at w.

The first-order condition for price implies that $\mathfrak{E}_{\mathsf{mon}}(p;w) = 1$ must hold at the monopoly equilibrium. Moreover, $\mathfrak{E}_{\mathsf{mon}}(p;w) > 1$ implies that the firm can increase profit by decreasing the price, and $\mathfrak{E}_{\mathsf{mon}}(p;w) < 1$ implies that the firm can increase profit by increasing the price.

A key observation is that, at a standard payment profile (p,w) we have

$$\begin{aligned} k'(p;w) &= D'(b(p;w)) \cdot b'(p;w) \\ &= D'(b(p;w)) \cdot \mathrm{d}\,\frac{p + c(mw - k(p;w))}{\mathrm{d}\,p} \\ &= D'(b(p;w)) \cdot \left(1 + \frac{-1}{m} \cdot (-k'(p;w))\right), \end{aligned} \tag{G.19}$$

where (G.19) holds because (p,w) is a standard payment profile. Solving for $k'(p;w)$ implies that, at any standard payment profile (p,w),

$$k'(p;w) = \frac{D'(b(p;w))}{1 - D'(b(p;w))/m}. \tag{G.20}$$

We now use (G.20) to rewrite (G.18) as

$$\mathfrak{E}_{\mathsf{mon}}(p;w) = \left(-\frac{D'(b(p;w))}{D(b(p;w))} \cdot \frac{1}{1 - D'(b(p;w))/m}\right) \cdot (p - w). \tag{G.21}$$

We will use the following definition to denote the first multiplicand on the right-hand side as a function of b: define

$$f_{\mathsf{mon}}(b) = \left(-\frac{D'(b)}{D(b)} \cdot \frac{1}{1 - D'(b)/m}\right).$$

Lemma G.15. *For $b > 1 - a/2$, the function $f_{\mathsf{mon}}(b)$ is increasing in b.*

Proof. We compute the closed-form expression for the derivative in file "fmon", which is

$$f'_{\mathsf{mon}}(b) = -\frac{2(a-1)a}{(a^2m - am + b - 1)^2} > 0.$$

This proves the claim. $\square$

G.2.2 Proof of Lemma G.14: $p_{\mathsf{duo}}(m) > p_{\mathsf{mon}}(m)$

Fix $m \in (\underline{m}, m')$. Let $(p_{\mathsf{duo}}, w_{\mathsf{duo}})$ denote the symmetric payment profile at the symmetric duopoly equilibrium. Also, let k denote the rate of customers joining firm 1 at the duopoly equilibrium. Also, let b_{duo} denote the aggregate cost that customers incur in the duopoly equilibrium.

Now, consider the setting with one firm; i.e., suppose firm 2 is excluded. There is a standard payment profile (p, w) such that $D(b(p; w)) = k$. That is, with price p and wage w, firm 1 would serve just as many customers as in the duopoly equilibrium. Similarly, for any positive $b < b(p; w)$, there are a price $p(b)$ and a wage $w(b)$ such that $D(b) = k(p(b); w(b))$ and, moreover, such that $(p(b), w(b))$ form a standard payment profile.

To complete the proof for this part, we will show that $\mathfrak{L}_{\mathsf{mon}}(p(b); w(b)) > 1$ for all $b \geq b_{\mathsf{duo}}$. This will imply that the aggregate cost incurred by customers in the monopoly equilibrium, namely b_{mon}, must be smaller than b_{duo}. Observe that by (G.8) and (G.12), the waiting cost incurred by customers is identical in both equilibria. Therefore, $b_{\mathsf{mon}} < b_{\mathsf{duo}}$ would imply that $p_{\mathsf{mon}} < p_{\mathsf{duo}}$. Hence, to complete this part, it remains to prove the following claim.

Claim G.16. $\mathfrak{L}_{\mathsf{mon}}(p(b); w(b)) > 1$ *for all* $b \geq b_{\mathsf{duo}}$.

Proof. We first recall the following equations from Section G.2.1.

$$\mathfrak{L}_{\mathsf{mon}}(p; w) = \left(-\frac{D'(b(p; w))}{D(b(p; w))} \cdot \frac{1}{1 - D'(b(p; w))/m}\right) \cdot (p - w),$$

and

$$f_{\mathsf{mon}}(b) = \left(-\frac{D'(b)}{D(b)} \cdot \frac{1}{1 - D'(b)/m}\right).$$

Let $g_{\mathsf{mon}}(b) = p(b) - w(b)$. Similarly, define

$$\mathfrak{L}_{\mathsf{mon}}(b) = \left(-\frac{D'(b)}{D(b)} \cdot \frac{1}{1 - D'(b)/m}\right) \cdot (p(b) - w(b)).$$

To prove the claim, we show that (i) $\mathfrak{L}_{\mathsf{mon}}(b_{\mathsf{duo}}) > 1$, and (ii) $\mathfrak{L}_{\mathsf{mon}}(b)$ is increasing in b for all $b > 1 - a/2$. This will prove the claim.

Part (i). Next, we show that $\mathfrak{L}_{\mathsf{mon}}(b_{\mathsf{duo}}) > 1$. First, recall the definition of adjusted price f(beta) in a duopoly equilibrium from (G.16): define the quantity

$$\mathfrak{L}_{\mathsf{duo}} = \left(-\frac{D_1(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}{D(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}\right) \cdot (p_{\mathsf{duo}} - w_{\mathsf{duo}}), \tag{G.22}$$

where recall that $D(p_1, w_1; p_2, w_2)$ denotes the rate of customers joining firm 1 in $\Sigma_{\mathbf{P}}$ under the payment profile $\mathbf{P} = ((p_1, w_1), (p_2, w_2))$. Furthermore, $D_1(\cdot, w_1; p_2, w_2)$ denotes the partial derivative of the function $D(\cdot, w_1; p_2, w_2)$ with respect to its first argument. The first-order condition of firm 1 with respect to price in the duopoly equilibrium, (G.1), implies that $\mathfrak{L}_{\mathsf{duo}} = 1$. Hence, to show that $\mathfrak{L}_{\mathsf{mon}}(b_{\mathsf{duo}}) > 1$, it suffices to show that $\mathfrak{L}_{\mathsf{mon}}(b_{\mathsf{duo}}) > \mathfrak{L}_{\mathsf{duo}}$. Equivalently, we will show that

$$\begin{aligned} \frac{\mathfrak{L}_{\mathsf{mon}}(b_{\mathsf{duo}})}{\mathfrak{L}_{\mathsf{duo}}} &= \frac{f_{\mathsf{mon}}(b_{\mathsf{duo}})}{f_{\mathsf{duo}}} \cdot \frac{p(b_{\mathsf{duo}}) - w(b_{\mathsf{duo}})}{p_{\mathsf{duo}} - w_{\mathsf{duo}}} > 1 \\ \Leftrightarrow \frac{f_{\mathsf{duo}}}{f_{\mathsf{mon}}(b_{\mathsf{duo}})} &< \frac{p(b_{\mathsf{duo}}) - w(b_{\mathsf{duo}})}{p_{\mathsf{duo}} - w_{\mathsf{duo}}} \end{aligned} \tag{G.23}$$

where f_{duo} denotes the first multiplicand on the right-hand side of (G.22). To show that the above inequality holds, we first compute a closed-form expression for $\frac{p(b_{\mathsf{duo}}) - w(b_{\mathsf{duo}})}{p_{\mathsf{duo}} - w_{\mathsf{duo}}}$. First, we recall that $(p(b), w(b))$ and $((p_{\mathsf{duo}}, w_{\mathsf{duo}}), (p_{\mathsf{duo}}, w_{\mathsf{duo}}))$ are standard payment profiles (respectively for the settings with one firm and two firms). Then, by definition,

$$c'(mw(b_{\mathsf{duo}}) - k) = c'(mw_{\mathsf{duo}} - 2k) = \frac{-1}{m}.$$

This implies that $mw(b_{\mathsf{duo}}) - k = mw_{\mathsf{duo}} - 2k$, and that $p(b) = p_{\mathsf{duo}}$ (because we must have $p(b) + c(mw(b_{\mathsf{duo}}) - k) = p_{\mathsf{duo}} + c(mw_{\mathsf{duo}} - 2k)$). These two equalities together imply that

$$\frac{p(b_{\mathsf{duo}}) - w(b_{\mathsf{duo}})}{p_{\mathsf{duo}} - w_{\mathsf{duo}}} = 1 + \frac{k}{m(p_{\mathsf{duo}} - w_{\mathsf{duo}})} = 1 - \frac{D_1(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}{m},$$

where the last equality holds by the firm's first-order condition with respect to price in the duopoly equilibrium(G.1). In file "duo-focp", we computed a closed-form expression for $D_1(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})$; plugging that expression in the above equality implies that

$$\frac{p(b_{\mathsf{duo}}) - w(b_{\mathsf{duo}})}{p_{\mathsf{duo}} - w_{\mathsf{duo}}} = 1 + \frac{(-1 + b_{\mathsf{duo}})(-1 - am + a^2 m + b_{\mathsf{duo}})}{m(-1 + a)a(-2 - am + a^2 m + 2b_{\mathsf{duo}})}.$$

This gives a closed-form expression in terms of b_{duo} for the right-hand side of (G.23). To prove that (G.23) holds, we also compute a closed-form expression in terms of b_{duo} for its left-hand side. This is done in file "bmon-vs-bduo". As shown there, comparing these two expressions implies that (G.23) always holds when $b_{\mathsf{duo}} > 1 - a/4$. The latter inequality always holds for sufficiently small $m' > \underline{m}$.

Part (ii). This step completes the proof of the claim by showing that $\mathfrak{L}_{\mathsf{mon}}(b)$ is increasing in b for all $b > 1 - a/2$. This is done in file "fmon", where we compute a closed-form solution for $\mathfrak{L}_{\mathsf{mon}}(b)$ and its derivative.

□

G.2.3 Proof of Lemma G.14: $w_{\mathsf{duo}}(m) > w_{\mathsf{mon}}(m)$

Our approach involves defining another planner, whom we call the double-monopolist, who owns both firms and whose goal is maximizing profit by posting a price and a wage (p, w). The optimal solution to the double-monopolist's problem is called the double-monopoly equilibrium.

We denote the monopolist planner by mon, and the double-monopolist planner by dm. For any planner $\mathsf{pl} \in \{\mathsf{mon}, \mathsf{dm}\}$, the optimal solution to the planner's problem is denoted by $(p_{\mathsf{pl}}, w_{\mathsf{pl}}, k_{\mathsf{pl}})$. For example, the optimal solution to the double-monopolist's problem is defined by $(p_{\mathsf{dm}}, w_{\mathsf{dm}}, k_{\mathsf{dm}})$. For $\mathsf{pl} = \mathsf{duo}$, $p_{\mathsf{duo}}, w_{\mathsf{duo}}$ respectively denote the price and wage at the symmetric duopoly equilibrium, and k_{duo} denotes the *total* rate of customers who request service at the duopoly equilibrium. Consistent with our usual notation, we use the variable b to denote $p + c(mw - k)$. For example, $b_{\mathsf{mon}} = p_{\mathsf{mon}} + c(mw_{\mathsf{mon}} - k_{\mathsf{mon}})$.

The proof involves two steps. In Step 1, we show that $w_{\mathsf{mon}} < w_{\mathsf{dm}}$, and in Step 2 we prove that $w_{\mathsf{dm}} < w_{\mathsf{duo}}$. A key fact that will be used in the proof is that the following equations hold:

$$c'(mw_{\mathsf{mon}} - k_{\mathsf{mon}}) = -1/m, \tag{G.24}$$

$$c'(mw_{\mathsf{duo}} - k_{\mathsf{duo}}) = -1/m, \tag{G.25}$$

$$c'(mw_{\mathsf{dm}} - k_{\mathsf{dm}}) = -1/m. \tag{G.26}$$

We have proved the first and second equations in Proposition A.3 and in Section F.1, respectively. The proof for the third equation is essentially identical to the proof for the first equation. We do not repeat the proof here.

Step 1. The proof is by contradiction. Suppose $w_{\text{mon}} > w_{\text{dm}}$. Then, by (G.24) and (G.26), we must have $k_{\text{mon}} > k_{\text{dm}}$. For each planner, namely $\mathsf{pl} \in \{\mathsf{mon}, \mathsf{dm}\}$, define the *adjusted price f(beta)* for that planner at payment profile (p, w) by

$$\mathfrak{L}_{\mathsf{pl}}(p; w) = -\frac{k'_{\mathsf{pl}}(p; w)}{k_{\mathsf{pl}}(p; w)} \cdot (p - w) = -\frac{D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p; w)) \cdot b'(p; w)}{D_{\mathsf{pl}}(b_{\mathsf{pl}}(p; w))} \cdot (p - w), \tag{G.27}$$

where the parameters in the above equation are defined as follows. For any planner pl, the function $k_{\mathsf{pl}}(p; w)$ denotes the rate of customers who request service as a function of the price, p, while holding wage fixed at w. Similarly, $b_{\mathsf{pl}}(p; w)$ denotes the aggregate cost that customers face as a function of p. The function $D_{\mathsf{pl}}(b)$ gives the rate of customers requesting service when the aggregate cost offered by the planner is b.

The first-order condition for the price implies that $\mathfrak{L}_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}) = 1$ should hold for all planners $\mathsf{pl} \in \{\mathsf{mon}, \mathsf{dm}\}$. To complete the proof in this step, we will show that $\mathfrak{L}_{\mathsf{dm}}(p_{\mathsf{dm}}; w_{\mathsf{dm}}) > 1$, which will be a contradiction. (This would mean that planner dm can increase profit by reducing the price, which would be a contradiction.)

First, observe that

$$\begin{aligned}
k'_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}) &= D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}})) \cdot b'(p_{\mathsf{pl}}; w_{\mathsf{pl}}) \\
&= D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}})) \cdot \frac{\mathrm{d}(p_{\mathsf{pl}} + c(mw_{\mathsf{pl}} - k_{\mathsf{pl}}))}{\mathrm{d}\, p} \\
&= D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}})) \cdot \left(1 + \frac{-1}{m} \cdot (-k'_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}))\right),
\end{aligned} \tag{G.28}$$

where (G.28) holds by (G.24),(G.25), and (G.26). Solving for $k'_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}})$ implies that

$$k'_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}) = \frac{D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}))}{1 - D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}))/m}. \tag{G.29}$$

We now use (G.29) to rewrite (G.27) as

$$\mathfrak{L}_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}) = \left(-\frac{D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}))}{D_{\mathsf{pl}}(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}))} \cdot \frac{1}{1 - D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}; w_{\mathsf{pl}}))/m}\right) \cdot (p_{\mathsf{pl}} - w_{\mathsf{pl}}). \tag{G.30}$$

Let us denote the first and second multiplicands on the right-hand side by f_{pl} and g_{pl}, respectively. We will show that $f_{\mathsf{dm}} \geq f_{\mathsf{mon}}$ and $g_{\mathsf{dm}} > g_{\mathsf{mon}}$, which will imply that $\mathfrak{L}_{\mathsf{dm}}(p_{\mathsf{dm}}; w_{\mathsf{dm}}) > \mathfrak{L}_{\mathsf{mon}}(p_{\mathsf{mon}}; w_{\mathsf{mon}})$, which is a contradiction. Proving $g_{\mathsf{dm}} > g_{\mathsf{mon}}$ is straightforward: we know that $w_{\mathsf{dm}} < w_{\mathsf{mon}}$ holds by assumption. On the other hand, because $k_{\mathsf{mon}} > k_{\mathsf{dm}}$ holds, then we should also have $b_{\mathsf{dm}} > b_{\mathsf{mon}}$. (G.24) and (G.26) then imply that $p_{\mathsf{dm}} > p_{\mathsf{mon}}$. Therefore, $g_{\mathsf{dm}} > g_{\mathsf{mon}}$.

To complete Step 1, it remains to show that $f_{\mathsf{dm}} \geq f_{\mathsf{mon}}$. To this end, first with slight abuse of notation we define the function $f_{\mathsf{dm}}(b)$ as follows

$$f_{\mathsf{dm}}(b) = \left(-\frac{D'_{\mathsf{dm}}(b)}{D_{\mathsf{dm}}(b)} \cdot \frac{1}{1 - D'_{\mathsf{dm}}(b)/m} \right).$$

To complete Step 1, it suffices to show that (i) $f_{\mathsf{dm}}(b)$ is an increasing function of b, and (ii) $f_{\mathsf{dm}}(b^*) \geq f_{\mathsf{mon}}$, where b^* is such that $D_{\mathsf{dm}}(b^*) = D_{\mathsf{mon}}(b_{\mathsf{mon}})$. Once these are proved, then $b_{\mathsf{dm}} > b_{\mathsf{mon}}$ would imply that $f_{\mathsf{dm}} \geq f_{\mathsf{mon}}$, which would conclude Step 1.

Part (i). $f_{\mathsf{dm}}(b)$ is an increasing function of b. This is shown in file "wage-dbm-vs-mon-1", where we compute a closed-form expression for $f'_{\mathsf{dm}}(b)$. In particular, we show that

$$f'_{\mathsf{dm}}(b) = \frac{2(a-1)am\left(a^2m - am + 4b - 4\right)}{(b-1)^2\left(-a^2m + am - 2b + 2\right)^2}$$

holds when $m > \underline{m}$ and m is sufficiently close to $\underline{m}$. We recall that $a = 1 - \sigma$ is in the unit interval, which means that the right-hand side is positive.

Part (ii). $f_{\mathsf{dm}}(b^*) \geq f_{\mathsf{mon}}$, where b^* is such that $D_{\mathsf{dm}}(b^*) = D_{\mathsf{mon}}(b_{\mathsf{mon}})$. This is shown in file "wage-dbm-vs-mon-2", where we compute b^* and $f_{\mathsf{dm}}(b^*)$ in terms of b_{mon}, and then compare $f_{\mathsf{dm}}(b^*)$ to f_{mon}, showing that the latter is not larger than the former. In particular, we compute

$$\begin{aligned}
b^* &= -\frac{a^2\sqrt{\frac{(b_{\mathsf{mon}}-1)^2}{(a-1)^2a^2}}}{\sqrt{2}} + \frac{a\sqrt{\frac{(b_{\mathsf{mon}}-1)^2}{(a-1)^2a^2}}}{\sqrt{2}} + 1, \\
f_{\mathsf{dm}}(b^*) &= \frac{2\sqrt{2}m}{(1 - b_{\mathsf{mon}})\left(\frac{\sqrt{2}(1-b_{\mathsf{mon}})}{(1-a)a} + m\right)} \\
f_{\mathsf{mon}} &= -\frac{2(a-1)am}{(b_{\mathsf{mon}} - 1)\left(a^2m - am + b_{\mathsf{mon}} - 1\right)}
\end{aligned}$$

Then, we compare the right-hand sides of the last two equations in file "wage-dbm-vs-mon-2" and show that the former is at least as large as the latter, whenever $a, b_{\mathsf{mon}} \in (0, 1)$. We spare the algebraic calculations here.

Step 2. In this step, we prove that $w_{\mathsf{dm}} < w_{\mathsf{duo}}$. Note that by (G.25) and (G.26), we have

$$w_{\mathsf{dm}} < w_{\mathsf{duo}} \Leftrightarrow k_{\mathsf{dm}} < k_{\mathsf{duo}}.$$

The proof is by contradiction. Suppose $w_{\mathsf{dm}} > w_{\mathsf{duo}}$, which also means $k_{\mathsf{dm}} > k_{\mathsf{duo}}$. Because the former implies $b_{\mathsf{dm}} < b_{\mathsf{duo}}$, and also because of (G.25) and (G.26), $p_{\mathsf{dm}} < p_{\mathsf{duo}}$ must hold as well.

We need two definitions before proceeding further. We say that a firm f *has a standard payment profile* in a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ if the condition

$$c'(mw_f - k) = \frac{-1}{m}$$

is satisfied.[18] Also, a *standard payment profile for the double-monopolist* is a profile that satisfies (G.26).

We first summarize the rest of the argument in Step 2. In the first part of the argument, we will show that the adjusted price f(beta) for the double-monopolist computed at a standard payment profile with aggregate cost b is increasing in b. In the second part, we will show that the payment profile used in the duopoly equilibrium is a standard payment profile for the double-monopolist, and that at this payment profile, the adjusted price f(beta) for the double-monopolist is larger than the adjusted price f(beta) for a firm in the duopoly equilibrium (which is just equal to 1). This will be a contradiction, since the adjusted price f(beta) for the double-monopolist computed at the double-monopoly equilibrium should equal 1.

To state this argument formally, we first write the adjusted price f(beta) for the double-monopolist at a standard payment profile as a function of the customers' aggregate cost, b. To this end, we use the notation $(p_{\mathsf{dm}}(b), w_{\mathsf{dm}}(b))$ to denote the double-monopolist's standard payment profile as a function of b, i.e., the payment profile $(p_{\mathsf{dm}}(b), w_{\mathsf{dm}}(b))$ is the (unique) standard payment profile under which the customers incur aggregate cost b. We now define

$$\begin{aligned}\mathfrak{L}_{\mathsf{dm}}(b) &= -\frac{k'_{\mathsf{dm}}(p_{\mathsf{dm}}(b); w_{\mathsf{dm}}(b))}{k_{\mathsf{dm}}(p_{\mathsf{dm}}(b); w_{\mathsf{dm}}(b))} \cdot (p_{\mathsf{dm}}(b) - w_{\mathsf{dm}}(b)) \\ &= -\frac{D_{\mathsf{dm}}'(b) \cdot b'(p_{\mathsf{dm}}(b))}{D_{\mathsf{dm}}(b)} \cdot (p_{\mathsf{dm}}(b) - w_{\mathsf{dm}}(b)). \qquad \text{(G.31)}\end{aligned}$$

Because the equilibrium payment profile is standard, we can use (G.30) to rewrite (G.31) as

[18] Intuitively, this conditon means that holding k_f fixed, firm f has chosen price and wage optimally.

follows:

$$\mathfrak{L}_{\mathsf{dm}}(b) = \left(-\frac{D_{\mathsf{dm}}{}'(b)}{D_{\mathsf{dm}}(b)} \cdot \frac{1}{1 - D_{\mathsf{dm}}{}'(b)/m}\right) \cdot (p_{\mathsf{dm}}(b) - w_{\mathsf{dm}}(b)). \tag{G.32}$$

There are two multiplicands on the right-hand side of (G.32). We denote these multiplicands as functions of b, with the first and second multiplicand denoted by $f(b), g(b)$, respectively. We will complete the proof by showing that (i) $\mathfrak{L}_{\mathsf{dm}}(b)$ is an increasing function of b, and (ii) $\mathfrak{L}_{\mathsf{dm}}(b_{\mathsf{duo}}) < 1$. These two facts together with $\mathfrak{L}_{\mathsf{dm}}(b_{\mathsf{dm}}) = 1$ will imply that $b_{\mathsf{dm}} > b_{\mathsf{duo}}$, which would imply that $k_{\mathsf{dm}} < k_{\mathsf{duo}}$. Hence, by (G.25) and (G.26), we must have $w_{\mathsf{dm}} < w_{\mathsf{duo}}$.

Part (i) $\mathfrak{L}_{\mathsf{dm}}(b)$ is an increasing function of b. It suffices to show that $f(b)$ and $g(b)$ are increasing in b. Recall that $g(b) = p_{\mathsf{dm}}(b) - w_{\mathsf{dm}}(b)$, and recall that $(p_{\mathsf{dm}}(b), w_{\mathsf{dm}}(b))$ is a standard payment profile. Hence, by (G.26), $w_{\mathsf{dm}}(b)$ is decreasing in b. Also, (G.26) implies that the $c(mw_{\mathsf{dm}}(b) - D_{\mathsf{dm}}(b))$ is a constant independent of b. Hence, $p_{\mathsf{dm}}(b)$ must be increasing in b. Therefore, $g(b)$ is increasing in b. File "fdm" shows that $f(b)$ is increasing in b as well, by computing its closed-form expression in terms of b and computing the derivative $f'(b)$. In particular, there we compute that

$$f'(b) = \frac{2(a-1)am\left(a^2m - am + 4b - 4\right)}{(b-1)^2\left(-a^2m + am - 2b + 2\right)^2} > 0,$$

where the inequality holds because $a, b \in (0,1)$.

Part (ii) $\mathfrak{L}_{\mathsf{dm}}(b_{\mathsf{duo}}) < 1$. First, recall from (G.22) that

$$\mathfrak{L}_{\mathsf{duo}} = \left(-\frac{D_1(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}{D(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}\right) \cdot (p_{\mathsf{duo}} - w_{\mathsf{duo}}),$$

and that $\mathfrak{L}_{\mathsf{duo}} = 1$ must hold. Hence, to prove this part, we will show that

$$\begin{aligned}\mathfrak{L}_{\mathsf{dm}}(b_{\mathsf{duo}}) &= \left(-\frac{D_{\mathsf{dm}}{}'(b_{\mathsf{duo}})}{D_{\mathsf{dm}}(b_{\mathsf{duo}})} \cdot \frac{1}{1 - D_{\mathsf{dm}}{}'(b_{\mathsf{duo}})/m}\right) \cdot (p_{\mathsf{dm}}(b_{\mathsf{duo}}) - w_{\mathsf{dm}}(b_{\mathsf{duo}})) \\ &< \left(-\frac{D_1(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}{D(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}\right) \cdot (p_{\mathsf{duo}} - w_{\mathsf{duo}}).\end{aligned} \tag{G.33}$$

To this end, first observe that $p_{\mathsf{dm}}(b_{\mathsf{duo}}) - w_{\mathsf{dm}}(b_{\mathsf{duo}}) = p_{\mathsf{duo}} - w_{\mathsf{duo}}$ must hold, because $(p_{\mathsf{dm}}(b_{\mathsf{duo}}), w_{\mathsf{dm}}(b_{\mathsf{duo}}))$ and $(p_{\mathsf{duo}}, w_{\mathsf{duo}})$ are standard payment profiles, respectively for the double-monopolist and for firm 1 in the duopoly equilibrium. Since, we must have that $D_{\mathsf{dm}}(b_{\mathsf{duo}}) = k_{\mathsf{duo}}$, then (G.25) and (G.26) imply that $p_{\mathsf{dm}}(b_{\mathsf{duo}}) = p_{\mathsf{duo}}$ and $w_{\mathsf{dm}}(b_{\mathsf{duo}}) = w_{\mathsf{dm}}$. Therefore, to prove that (G.33) holds, it suffices to show that

$$\left(-\frac{D_{\mathsf{dm}}{}'(b_{\mathsf{duo}})}{D_{\mathsf{dm}}(b_{\mathsf{duo}})} \cdot \frac{1}{1 - D_{\mathsf{dm}}{}'(b_{\mathsf{duo}})/m}\right) < \left(-\frac{D_1(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}{D(p_{\mathsf{duo}}, w_{\mathsf{duo}}; p_{\mathsf{duo}}, w_{\mathsf{duo}})}\right) \tag{G.34}$$

We prove this inequality in file "wage-duo-vs-dbm", where we compute the closed-form expressions for its left- and right-hand side. In particular, we compute that the left-hand side of the above inequality is equal to

$$\frac{2(1-a)am}{(1-b_{\mathsf{duo}})\left(a^2m - am + 2b_{\mathsf{duo}} - 2\right)}.$$

and that the right-hand side is equal to

$$\frac{2\left(a^2m - am + b_{\mathsf{duo}} - 1\right)}{(1-b_{\mathsf{duo}})\left(a^2m - am + 2b_{\mathsf{duo}} - 2\right)}.$$

We then show that the left-hand side is smaller than the right-hand side when $a, b_{\mathsf{duo}} \in (0, 1)$.

G.2.4 Proof of Lemma G.14: $u^W_{\mathsf{duo}}(m) > u^W_{\mathsf{mon}}(m)$ and $u^C_{\mathsf{duo}}(m) < u^C_{\mathsf{mon}}(m)$

Recall (4.2) from Section 4.2 which computes the workers average welfare for the case of uniform distribution as

$$u^W(m) = \frac{1}{2} \cdot \left(w(m) + \frac{k(m)}{m}\right), \tag{G.35}$$

where $k(m), w(m)$ denote the equilibrium values for the rate of customers joining and for wage in a monopoly equilibrium. Note that this equality also holds for a symmetric duopoly equilibrium, where $k(m)$ would denote the total rate of customers requesting service at the duopoly equilibrium and $w(m)$ would denote wage.

Therefore, to prove $u^W_{\mathsf{duo}}(m) > u^W_{\mathsf{mon}}(m)$, it suffices to show that $w_{\mathsf{duo}} > w_{\mathsf{mon}}$ and $k_{\mathsf{duo}} > k_{\mathsf{mon}}$. We proved the former inequality in the previous section. Also, observe that the former inequality imples the latter: This holds because, by (A.1) and (G.8), we have

$$c'(mw_{\mathsf{mon}} - k_{\mathsf{mon}}) = c'(mw_{\mathsf{duo}} - k_{\mathsf{duo}}) = \frac{-1}{m}.$$

Hence, it remains to prove that $u^C_{\mathsf{duo}}(m) < u^C_{\mathsf{mon}}(m)$. For when $m \in (\underline{m}, m')$, we compute the average welfare of customers who request service as a function of the aggregate cost that they incur, namely b. This is done in file "customer-ave-welfare", where we compute that, in a monopoly equilibrium or a symmetric duopoly equilibrium, the average welfare is $\frac{1-b}{3}$, where b would denote the aggregate cost offered to customers by a firm. (G.35) together with $p_{\text{mon}} < p_{\text{duo}}$ imply that $b_{\text{mon}} < b_{\text{duo}}$, which proves the claim since $\frac{1-b}{3}$ is decreasing in b.

G.3 Proving that local equilibria are global equilibria

In the previous step of the proof (Section G.2) we showed that for any $m \in (\underline{m}, m')$, there exists a unique symmetric local duopoly equilibrium at m. In this step, the last step of the proof, we show that there exists $m'' \in (\underline{m}, m')$ such that any symmetric local equilibrium in the interval $(\underline{m}, m'')$ is also a (global) duopoly equilibrium. Setting $\hat{m} = m''$ will then prove the claim of the theorem.[19] We do not fix the value of m'' in advance; this value will be set in the course of the proof.

We need one definition before presenting the proof. We say that a firm f *has a standard payment profile* in the subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ if the condition

$$c'(mw_f - k) = \frac{-1}{m}$$

is satisfied. Intuitively, this condition means that holding k_f fixed, firm f has chosen price and wage optimally. (Recall the proof of Claim G.13 where we saw that if $c'(mw_f - k) \neq \frac{-1}{m}$, then firm f can choose a different price and wage to increase its profit, while serving the same rate of customers as before.)

Given a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, we say that a deviation $\overline{\mathbf{P}}_f$ is a *standard deviation* for firm f if $\overline{\mathbf{P}}_f$ is a standard payment profile for firm f in $\Sigma_{\overline{P}}$, where $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$. This definition is important in proving that any local duopoly equilibrium is also a (global) duopoly equilibrium: to prove this, without loss of generality, it sufficies to prove that the standard deviations for firm f cannot increase its profit. More precisely, let $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ denote the payment profile after the deviation of firm f. The idea is that if $\overline{\mathbf{P}}_f$ is not a standard deviation for f, then firm f can choose a standard deviation $\overline{\overline{\mathbf{P}}}_f$ which increases her profit more than the deviation $\overline{P}_f$. This claim holds essentially by the same argument that proves Claim G.13. This significantly simplifies the analysis, as it helps to interpret the game played between the firms as a game in which the firms compete by offering aggregate

[19] We believe that any symmetric local duopoly equilibrium is also a (global) duopoly equilibrium, however, proving the claim for the interval $(\underline{m}, m'')$ seems to be significantly simpler.

costs to customers, rather than offering a price to customers and a wage to workers. The details will become clear in the course of this proof.

Fix m, and let the symmetric local duopoly equilibrium at m be denoted by $\Sigma = (\mathbf{P}, \mathbf{A})$, with $\mathbf{P} = ((p_1, w_1), (p_2, w_2))$ and customer composition $\mathbf{k} = (k_1, k_2)$. The duopoly equilibrium is symmetric, and hence we let $p = p_1 = p_2$, $w = w_1 = w_2$, and $k_1 = k_2 = k/2$. Therefore, the total rate of customers that request service in Σ is k. As usual, we use the variable k_f to denote the rate of customers who join firm f in Σ; so, $k_f = k/2$. Also, we let b_f denote the aggregate cost offered to customers at firm f.

Suppose that firm 1 makes a deviation after which the aggregate cost that it offers to customers changes to $b_1^{\#} = b_1 - \delta$. We prove that such a deviation does not increase the profit of firm 1, for any positive or negative δ. We can assume that the deviation made by firm 1 is a standard deviation; this is without loss of generality: if the deviation is not standard, then firm 1 can choose a new deviation which is standard and increases her profit more than the non-standard deviation. (This follows from the same argument in the proof of Claim G.13).

We present the proof for the case of $\delta > 0$ first. The case of $\delta < 0$ is similar, and will be proved at the end. Let $p_1^{\#}, w_1^{\#}, k_1^{\#}$ respectively denote the price and wage offered by firm 1, and the mass of customers that join firm 1 after her deviation. Also, let $k_2^{\#}$ denote the total mass of customers who join firm 2 after firm 1's deviation, and let $k^{\#} = k_1^{\#} + k_2^{\#}$.

Next, we will show that all the variables $p_1^{\#}, w_1^{\#}, k_1^{\#}, k_2^{\#}, k^{\#}$ are uniquely determined once δ is fixed. (Later in the proof, we will use this property to write these variables as a function of δ.) It is straightforward to compute $k_1^{\#}$ as a function of $b_1^{\#}$: by our choice of m', $k_1^{\#}$ is the area of the shaded triangle in Figure 11, which could clearly be written as a function of $b_1^{\#}$. We use the market-clearing equation to show why $k_2^{\#}$ is uniquely determined by δ:

$$k_1^{\#} + \hat{k}_2 = k_1^{\#} + D(p_2 + c(mw_2 - k_1^{\#} - \hat{k}_2), b_1^{\#}).$$

In the above equation, we have used the variable $\hat{k}_2$ to take the place of $k_2^{\#}$. Observe that the left-hand side of the above equation is strictly increasing in $\hat{k}_2$, but its right-hand side is decreasing in $\hat{k}_2$. There exists a unique value of $\hat{k}_2$ that solves the above equation, which we denoted by $k_2^{\#}$. This also implies that $k^{\#}$ is uniquely determined by δ, since $k^{\#} = k_1^{\#} + k_2^{\#}$.

Because $c'(mw_1^{\#} - k^{\#}) = \frac{-1}{m}$, then $w_1^{\#}$ is also uniquely determined by δ, and so is $p_1^{\#}$, because $b_1^{\#} = p_1^{\#} + c(mw_1^{\#} - k^{\#})$. Moreover,

$$c'(mw_1^{\#} - k^{\#}) = c'(mw_1 - k) = \frac{-1}{m} \tag{G.36}$$

implies that $p_1 - p_1^{\#} = \delta$.

The profit of firm 1 before and after the deviation, respectively, is

$$\Pi_1 = k_1 \cdot (p_1 - w_1),$$
$$\Pi_1^{\#} = k_1^{\#} \cdot (p_1^{\#} - w_1^{\#}).$$

Next, we write $\Pi_1^{\#}$ in a slightly different form:

$$\Pi_1^{\#} = \left(k_1 + \Delta_{[k_1]}\right) \cdot \left(p_1 - w_1 - \Delta_{[p_1]} - \Delta_{[w_1]}\right),$$

where

$$\Delta_{[k_1]} = k_1^{\#} - k_1,$$
$$\Delta_{[p_1]} = p_1 - p_1^{\#} = \delta,$$
$$\Delta_{[w_1]} = w_1^{\#} - w_1.$$

Later on, we will see that

$$\Delta_{[k_1]}, \Delta_{[p_1]}, \Delta_{[w_1]} > 0.$$

In terms of these new variables, we can now write the inequality $\Pi_1 \geq \Pi_1^{\#}$ as

$$(k_1 + \Delta_{[k_1]}) \cdot (\Delta_{[w_1]} + \Delta_{[p_1]}) \geq \Delta_{[k_1]} \cdot (p_1 - w_1). \tag{G.37}$$

We will show that the above inequality holds for all $\delta > 0$.

We start by providing a lower-bound for $\Delta_{[w_1]}$. First, we observe that $w_1^{\#} = \frac{c'^{-1}(-1/m)+k^{\#}}{m}$ holds by (G.36). To provide the promised lower-bound we will show that $k^{\#}$, when written as a function of δ, is increasing and convex in δ. After proving this claim, we use the derivative of $k^{\#}$ with respect to δ at point $\delta = 0$ to compute a lower-bound for $w_1^{\#}$, which will turn into a lower-bound for $\Delta_{[w_1]}$.

To make this argument precise, we use the notation $k^{\#}(\delta)$ to denote $k^{\#}$ as a function of δ. Similarly, we define functions $b_1^{\#}(\delta), b_2^{\#}(\delta)$ to denote the values of $b_1^{\#}$ and $b_2^{\#}$ as functions of δ. Recall that $b_1^{\#} = b_1 - \delta$. Define $\bar{\delta}_m = 1 - c(m)$. Hence, the interval $[0, \bar{\delta}_m]$ includes all possible values that δ can belong to, when firm 1 earns a positive profit after deviation.

Claim G.17. *m'' can be chosen such that, for any $m \in (\underline{m}, m'')$, $k^{\#}(\delta)$ is increasing and convex over $[0, \bar{\delta}_m]$.*

Proof. In file "ksharp-increasing" we compute $\frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}$ by implicit differentiation from the market-clearing equation with respect to δ. We compute

$$\frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta} = -\frac{1 - b_1^{\#}(\delta)}{(a-1)a - (b_2^{\#}(\delta) - 1) \cdot c' \left(m w_2 - k^{\#}(\delta)\right)} > 0,$$

where the inequality holds because $0 \le b_1^{\#}(\delta) < 1$ and $0 < b_2^{\#}(\delta) \le 1$ hold for all $\delta \in [0, \bar{\delta}]$.

To prove the convexity claim, we compute $\frac{\mathrm{d}^2\, k^{\#}(\delta)}{(\mathrm{d}\, \delta)^2}$ in file "ksharp-convex". The key point is that, as we compute there,

$$\lim_{\delta \to 0} \frac{\mathrm{d}^2\, k^{\#}(\delta)}{(\mathrm{d}\, \delta)^2} = \frac{\mathrm{d}^2\, k^{\#}(\delta)}{(\mathrm{d}\, \delta)^2}\Big|_{\delta=0} = \frac{1}{a - a^2} > 0, \tag{G.38}$$

holds for any $m > \underline{m}$. First of all, this implies the convexity of $k^{\#}(\delta)$ at $\delta = 0$. Second, from the closed-form expression that we compute for $\frac{\mathrm{d}^2\, k^{\#}(\delta)}{(\mathrm{d}\, \delta)^2}$ in file "ksharp-convex", we observe that $\frac{\mathrm{d}^2\, k^{\#}(\delta)}{(\mathrm{d}\, \delta)^2}$ is continuous in (m, δ) at all points (m, δ) for δ sufficiently close to 0. This fact together with (G.38) imply that there exists $\zeta > 0$ independent of m such that at all $m \in (\underline{m}, m')$ and $\delta \in (0, \zeta)$, we have $\frac{\mathrm{d}^2\, k^{\#}(\delta)}{(\mathrm{d}\, \delta)^2} > 0$. The proof then would be complete if we choose $m'' < m$ such that $\bar{\delta}_m < \zeta$ for all $m \in (\underline{m}, m'')$. To this end, recall that $\bar{\delta}_m = 1 - c(m)$. We choose m'' such that $1 - c(m'') < \zeta$, which would also mean that $\bar{\delta}_m < \zeta$. $\square$

In the rest of the proof, we prove (G.37) by proving a stronger version of it. As we discussed earlier, this stronger version is derived by writing a lower-bound for $\Delta_{[w_1]} = \frac{k^{\#} - k}{m}$. Using Claim G.17, we can write

$$\Delta_{[w_1]} = \frac{k^{\#}(\delta) - k^{\#}(0)}{m} \ge \frac{\delta}{m} \cdot \left(\frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0} \right) \tag{G.39}$$

We now write a stronger version of (G.37) using the above expression, while dividing both sides of (G.37) by $\Delta_{[k_1]}$:

$$\left(1 + \frac{k_1}{\Delta_{[k_1]}}\right) \cdot \left(\left(\frac{\delta}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0} \right) + \delta \right) \ge p_1 - w_1. \tag{G.40}$$

Define $\epsilon = \frac{\delta}{1-a}$. In order to simplify (G.40), we also use the fact that

$$k_1 = \frac{xy}{2}, \tag{G.41}$$

$$\Delta_{[k_1]} = \epsilon y + \frac{\epsilon^2 y}{2x}, \tag{G.42}$$

where $x = 1 - \frac{b_1 - a}{1-a}$, $y = \frac{1-b_1}{a}$, and $a = 1 - \sigma$, as illustrated in Figure 11 (This figure illustrates customers' valuations using the unit square, as discussed before in Figure 10). Computing the value of $\Delta_{[k_1]}$ in terms of x, y, ϵ is straightforward, as shown in Figure 12.

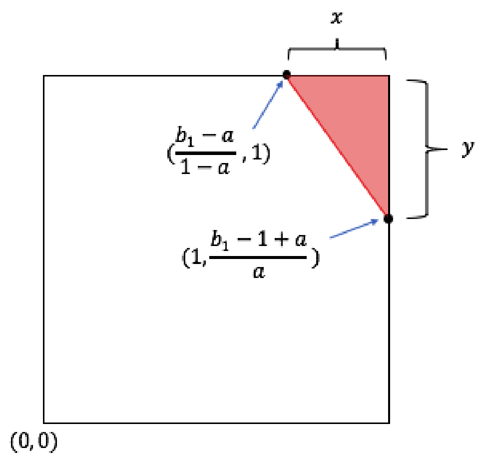

Figure 11: The shaded area represents the customers who join firm 1.

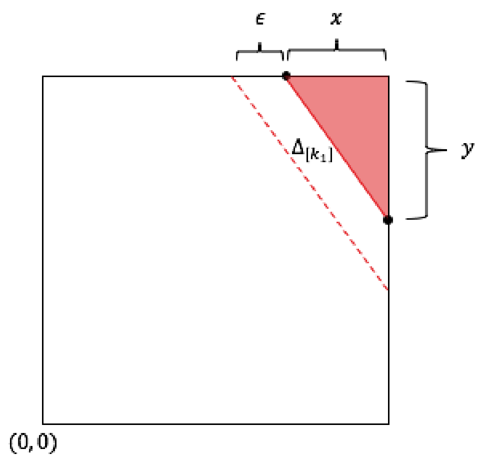

Figure 12: $\Delta_{[k_1]}$ is the area of the trapezoid, and can be written in terms of x, y, ϵ.

We now can rewrite (G.40) in the following way

$$\left(1+\frac{xy/2}{\epsilon y+\frac{\epsilon^2 y}{2x}}\right)\cdot\left(\left(\frac{\delta}{m}\cdot\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right)+\delta\right)\geq p_1-w_1,$$

which could be simplified to

$$\left(\epsilon+\frac{x}{2+\frac{\epsilon}{x}}\right)\cdot\left(\left(\frac{1-a}{m}\cdot\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right)+1-a\right)\geq p_1-w_1. \tag{G.43}$$

To prove that the above inequality holds, we observe that the the first multiplicand on the left-hand side is a function of ϵ, where as the second multiplicand is not. Therefore, to prove (G.43), it suffices to prove the following: (i) The first multiplicand is an increasing function of ϵ, and (ii) Equation (G.43) holds at $\epsilon=0$. These are proved in Steps (i) and (ii), respectively.

Step (i) We compute the derivative of the first multiplicand on the left-hand side of (G.43),

$$\frac{\mathrm{d}\left(\epsilon+\frac{x}{2+\frac{\epsilon}{x}}\right)}{\mathrm{d}\,\epsilon}=1-\frac{x^2}{(\epsilon+2x)^2}>0,$$

which is always positive. This completes step (i).

Step (ii) First, we rewrite (G.43) as follows.

$$\left(\delta+\frac{x(1-a)}{2+\frac{\delta}{(1-a)x}}\right)\cdot\left(\left(\frac{1}{m}\cdot\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right)+1\right)\geq p_1-w_1. \tag{G.44}$$

To prove (G.43) for $\epsilon=0$, we prove its equivalent (G.44) for $\delta=0$. Intuitively, (G.44) holds because we assumed the given subgame equilibrium is a *local* duopoly equilibrium, and therefore for sufficiently small $\delta>0$, firm 1's deviation should not increase her profit. More precisely, we demonstrate that when $\delta=0$, (G.44) in fact coincides with firm's first-order condition that ensures no (local) *standard* deviation is beneficial to the firm. To see why this holds, observe that when $\delta=0$, then (i) the term $\left(\delta+\frac{x(1-a)}{2+\frac{\delta}{(1-a)x}}\right)$ in (G.44) is just equal to

$$\frac{k^{\#}(\delta)}{\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}},$$

and (ii) the term $\left(\left(\frac{1}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}\right) + 1\right)$ is equal to

$$-\frac{\mathrm{d}\, p^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0} + \frac{\mathrm{d}\, w^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}.$$

This complete the proof for the case of $\delta > 0$. It remains to address the case of $\delta < 0$. The proof for this case is almost identical. We state the proof, skipping the steps that are identical to the previous proof. We use the same variables that we used in the previous proof. The variables $\Delta_{[k]}, \Delta_{[k_1]}, \Delta_{[p_1]}, \Delta_{[w_1]}$, are defined in a way that they are all positive. Also, we define $\epsilon = \frac{-\delta}{1-a}$ so that it is a positive quantity. To prove that firm 1's profit after deviation does not increase, we then have to show that

$$(k_1 - \Delta_{[k_1]}) \cdot (\Delta_{[w_1]} + \Delta_{[p_1]}) \leq \Delta_{[k_1]} \cdot (p_1 - w_1).$$

(This is the counterpart for (G.37)) Dividing both sides by $\Delta_{[k_1]}$, the above inequality can be written as

$$\left(\frac{k_1}{\Delta_{[k_1]}} - 1\right) \cdot (\Delta_{[w_1]} + \Delta_{[p_1]}) \leq p_1 - w_1.$$

Following the same proof steps as before, we can get the counterpart for inequality (G.43) as

$$\left(\frac{xy/2}{\epsilon y - \frac{\epsilon^2 y}{2x}} - 1\right) \cdot \left(\left(\frac{1-a}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}\right) + 1 - a\right) \leq p_1 - w_1,$$

or equivalently,

$$\left(\frac{x}{2 - \frac{\epsilon}{x}} - \epsilon\right) \cdot \left(\left(\frac{1-a}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}\right) + 1 - a\right) \leq p_1 - w_1. \tag{G.45}$$

We prove the above inequality in two steps: (i) We show that the first multiplicand is decreasing in ϵ, and (ii) We prove the inequality for $\epsilon = 0$. For step (i), we compute the derivative of the first multiplicand with respect to ϵ,

$$\frac{\mathrm{d}\left(\frac{x}{2-\frac{\epsilon}{x}} - \epsilon\right)}{\mathrm{d}\, \epsilon} = \frac{x^2}{(\epsilon - 2x)^2} - 1 < 0,$$

where the inequality holds since $\epsilon < x$. (Note that $\delta < 0$ implies that $\epsilon < x$) The proof for step (ii) is the same as the proof for its counterpart in the case of $\delta > 0$.

H Proof of Theorem 6.5: throughput maximizing firm

This section presents the proof of Theorem 6.5 for the case of throughput-maximizing firm. For brevity, in this section we call a throughput-maximizing monopoly equilibrium simply a monopoly equilibrium. Similarly, we call a throughput-maximizing duopoly equilibrium simply a duopoly equilibrium. First, we note that the cutoff characterization of steady-state subgame equilibria, as given in Section F.2, still applies. Thereby, Lemma F.6 holds as well.

First, we prove the existence and uniqueness of the equilibria. For the case of monopoly equilibrium, the proof was given in Section C, for when m is sufficiently close to $\underline{m} = \frac{1}{-c'(0)F'(0)}$. For the case of duopoly equilibrium, the proof is as follows.

We choose $\hat{m}$ small enough so that for any $m \in (\underline{m}, \hat{m})$, in any monopoly equilibrium, the aggregate cost that customers incur is at least $\frac{1+\sigma}{2}$. While larger m can be chosen, this particular choice makes the proof simpler.

With slight abuse of notation, define $\mathcal{D}(p_1, p_2) = D((p_1, p_1), (p_2, p_2))$. (Recall the definition of the right-hand side from Definition G.3.) This function determines the rate of costumers who request service from firm 1 if it posts price and wage equal to p_1 when firm 2 posts price and wage equal to p_2. Also, define $\mathcal{D}(p) = \mathcal{D}(p, p)$;

Let $p^* = \max_{p\in\mathbb{R}_+} \mathcal{D}(p)$. In file "msm-duo-foc", simplify the equation $\mathcal{D}'(p^*) = 0$ and show that it is equivalent to

$$mc'(mp^* - k^*) = -1, \tag{H.1}$$

where $k^* = 2\mathcal{D}(p^*, p^*)$ is the rate of customers who request service when both firms post a price and a wage equal to p^*. We will show that the payment profile

$$\mathbf{P} = ((p^*, p^*), (p^*, p^*)) \tag{H.2}$$

induces a duopoly equilibrium. If such a duopoly equilibrium exists, its uniqueness is guaranteed by Proposition 6.2.

To this end, we first show that $\mathbf{P}$ induces a local duopoly equilibrium. In file "msm-duo-foc", we simplify the first-order condition of firm 1 in a symmetric local duopoly equilibrium, i.e., $\mathcal{D}_1(p,p) = 0$, and show that this condition holds iff

$$mc'(mp - k) = -1, \tag{H.3}$$

where $k = 2\mathcal{D}(p,p)$. Hence, by (H.1), the payment profile $\mathbf{P} = ((p^*, p^*), (p^*, p^*))$, is a local duopoly equilibrium, as it satisfies (H.3). The proof for showing that $\mathbf{P}$ induces a global duopoly equilibrium is given next, in Section H.1. After that, in Section H.2 we complete the proof of the theorem by showing the uniqueness of symmetric local duopoly equilibrium and comparing the price and the customers' average welfare in duopoly and monopoly equilibria.

H.1 Proving that local equilibria are global equilibria

We show that, for $m \in (\underline{m}, m')$, any symmetric local equilibrium is also a (global) duopoly equilibrium. We need some definitions before presenting the proof. We typically use p_f to denote the price and wage posted by firm f and k_f to denote the rate of customers served by firm f. We recall that we say a firm f has a *standard* payment profile in the subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ if the condition

$$c'(mp_f - k) = \frac{-1}{m}$$

is satisfied.

Given a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, we say that a deviation $\overline{\mathbf{P}}_f$ is a *standard deviation* for firm f if $\overline{\mathbf{P}}_f$ is a standard payment profile for firm f in $\Sigma_{\overline{P}}$, where $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$. This definition is important in proving that any local duopoly equilibrium is also a (global) duopoly equilibrium: to prove this, without loss of generality, it suffices to prove that no standard deviation for firm f can increase the rate of customers served by that firm. More precisely, let $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ denote the payment profile after the deviation of firm f. If $\overline{\mathbf{P}}_f$ is not a standard deviation for f, then firm f can choose a standard deviation $\overline{\overline{\mathbf{P}}}_f$ which weakly increases the rate of its customers compared to $\overline{P}_f$. This is proved in the next claim.

Claim H.1. *Suppose that $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ is such that $\overline{\mathbf{P}}_f$ is not a standard payment profile for firm f in $\Sigma_{\overline{\mathbf{P}}}$. Then, a payment profile $\overline{\overline{\mathbf{P}}} = (\overline{\overline{\mathbf{P}}}_f, \mathbf{P}_{-f})$ exists such that firm f serves a weakly higher rate of customers in $\Sigma_{\overline{\overline{\mathbf{P}}}}$ than in $\Sigma_{\overline{\mathbf{P}}}$ and, furthermore, $\overline{\overline{\mathbf{P}}}_f$ is a standard payment profile for firm f in $\Sigma_{\overline{\overline{\mathbf{P}}}}$.*

Proof. Let $\overline{\overline{\mathbf{P}}}_f$ be chosen such that firm f servers the highest possible rate of customers in

$\Sigma_{\overline{\overline{\mathbf{P}}}}$. Recall a firm's first-order condition which sets the derivative of the rate of its customers with respect to its price to 0. Also, recall that this first-order condition (when simplified) is the same as (H.3). Hence, it must hold that

$$c'(m\overline{\overline{p}}_f - \overline{\overline{k}}) = \frac{-1}{m},$$

where $\overline{\overline{p}}_f$ denotes the price of firm f and $\overline{\overline{k}}$ denotes the total rate of customers served in $\Sigma_{\overline{\overline{\mathbf{P}}}}$. Hence, $\overline{\overline{\mathbf{P}}}_f$ is a standard profile for firm f in $\Sigma_{\overline{\overline{\mathbf{P}}}}$. This concludes the proof. $\square$

Fix m, and let the symmetric local duopoly equilibrium at m be denoted by $\Sigma = (\mathbf{P}, \mathbf{A})$, with $\mathbf{P} = ((p_1, w_1), (p_2, w_2))$ and customer composition $\mathbf{k} = (k_1, k_2)$. The duopoly equilibrium is symmetric, and hence we let $p = p_1 = p_2$, $w = w_1 = w_2$, and $k_1 = k_2 = k/2$. Therefore, the total rate of customers that are served in Σ is k. As usual, we let b_f denote the aggregate cost offered to customers at firm f.

Suppose that firm 1 makes a deviation after which the aggregate cost that it offers to customers changes to $b_1^\# = b_1 - \delta$. We prove that such a deviation does not increase the rate of customers who are served by firm 1, for any positive or negative δ. By (H.1), we can assume that the deviation made by firm 1 is a standard deviation.

First, observe that if $\delta < 0$, the rate of customers who join firm 1 will decrease. Hence, suppose $\delta > 0$. Let $p_1^\#, k_1^\#$ respectively denote the price (and wage) offered by firm 1, and the rate of customers that join firm 1 after her deviation. Also, let $k_2^\#$ denote the total rate of customers that join firm 2 after firm 1's deviation, and let $k^\# = k_1^\# + k_2^\#$.

Next, we will show that all the variables $p_1^\#, k_1^\#, k_2^\#, k^\#$ are uniquely determined once δ is fixed. (Later in the proof, we will use this property to write these variables as a function of δ.) It is straight-forward to compute $k_1^\#$ as a function of $b_1^\#$: by our choice of m', $k_1^\#$ is the area of the shaded triangle in Figure 11, which could clearly be written as a function of $b_1^\#$. We use the market-clearing equation to show why $k_2^\#$ is uniquely determined by δ:

$$k_1^\# + \hat{k}_2 = k_1^\# + D(p_2 + c(mp_2 - k_1^\# - \hat{k}_2), b_1^\#),$$

where we recall the definition of the function D (the firm's demand as a function of the aggregate costs offered by both firms) from Section F.1. In the above equation, we have used the variable $\hat{k}_2$ to take the place of $k_2^\#$. Observe that the left-hand side of the above equation is strictly increasing in $\hat{k}_2$, but its right-hand side is decreasing in $\hat{k}_2$. Hence, there exists a unique value of $\hat{k}_2$ that solves the above equation, which we called $k_2^\#$. This also implies that $k^{\#}$ is uniquely determined by δ, since $k^{\#} = k_1^{\#} + k_2^{\#}$.

Since the deviation of firm f is a standard deviation, then $c'(mp_1^{\#} - k^{\#}) = \frac{-1}{m}$, which means that $p_1^{\#}$ is also uniquely determined by δ. Moreover,

$$c'(mp_1^{\#} - k^{\#}) = c'(mp_1 - k) = \frac{-1}{m}$$

implies that $p_1 - p_1^{\#} = \delta$.

We use the notation $k^{\#}(\delta)$ to denote $k^{\#}$ as a function of δ. Similarly, we define functions $b_1^{\#}(\delta), b_2^{\#}(\delta)$ to denote the values of $b_1^{\#}$ and $b_2^{\#}$ as functions of δ.

Recall that $b_1^{\#} = b_1 - \delta$. Let $\bar{\delta} = 1 - c(m)$. Hence, the interval $[0, \bar{\delta}]$ includes all possible values that δ can take (i.e., the values under which a non-zero rate of customers join firm 1).

Claim H.2. *$k^{\#}(\delta)$ is increasing over $[0, \bar{\delta}]$.*

Proof. In file "msm-ksharp-increasing" we compute $\frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}$ by implicit differentiation from the market-clearing equation with respect to δ. We compute

$$\frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta} = -\frac{1 - b_1^{\#}(\delta)}{(a-1)a - (b_2^{\#}(\delta) - 1) \cdot c'\left(mp_2 - k^{\#}(\delta)\right)} > 0,$$

where the inequality holds because $0 \leq b_1^{\#}(\delta) < 1$ and $0 \leq b_2^{\#}(\delta) < 1$ hold for all $\delta \in [0, \bar{\delta}]$. $\square$

Because $k^{\#}(\delta)$ is increasing by the above claim, and since $\delta > 0$, the total rate of served customers (by either of the firms) should increase after the deviation of firm 1. On the other hand, we have

$$c'(mp_1^{\#} - k^{\#}) = c'(mp_1 - k) = \frac{-1}{m},$$

which means that if $k^{\#} > k$, then we must have $p_1^{\#} > p_1$. But this contradicts $\delta > 0$. Hence, there is no deviation for firm 1 that can increase the rate of customers that it serves at the local duopoly equilibrium. Hence the local equilibrium is a symmetric duopoly equilibrium.

H.2 Uniqueness of the local equilibrium and the comparative statics

The proof for uniqueness of local symmetric duopoly equilibrium is given in file "msm-local-uniqueness". There, we show that when $m \in (\underline{m}, \hat{m})$, the system of equations given by the firm's first-order condition (given by (H.3)), and the market-clearing condition has a unique solution $(p(m), k(m))$ satisfying $\lim_{m\to m_0} p(m) = 0$.

To complete the proof of Theorem 6.5 for the case of throughput maximizing firm, we will compare the price and the customers' average welfare in duopoly and monopoly equilibria. Consider a monopoly throughput-maximizing equilibrium, induced by a price and wage equal to p_{mon}, posted by firm 1. Let k_{mon}, denote the rate of customers served in the monopoly equilibrium. Also, consider the (local) duopoly equilibrium induced by the payment profile $\mathbf{P}$. (Recall the definition of $\mathbf{P}$ from (H.2).) Let k_{duo} denote the total rate of customers served in that duopoly equilibrium. Note that the rate of customers served by each firm in the duopoly equilibrium is therefore $k_{\text{duo}}/2$.

We will show that when $m \in (\underline{m}, \hat{m})$, then (i) $k_{\text{duo}}/2 < k_{\text{mon}} < k_{\text{duo}}$, and that (ii) $p_{\text{mon}} < p_{\text{duo}}$. If (i) holds, then the rate of customers served by firm 1 is larger in the monopoly equilibrium than in the duopoly equilibrium, and therefore, by the definition of the customers' utilities, the customers' average welfare is also higher in the monopoly equilibrium, i.e., $u^C_{\text{duo}}(m) < u^C_{\text{mon}}(m)$. Therefore, proving (i) and (ii) would complete the proof of the theorem.

Part (i) $k_{\text{duo}}/2 < k_{\text{mon}} < k_{\text{duo}}$. First, observe that $k_{\text{duo}}/2 \geq k_{\text{mon}}$ gives a contradiction: if it holds, then in a monopoly equilibrium firm 1 can post a price of p_{duo}, and serve a rate of customers strictly larger than k_{mon}, which would be a contradiction. Therefore, $k_{\text{duo}}/2 < k_{\text{mon}}$.

Next, we show that $k_{\text{mon}} < k_{\text{duo}}$. The proof is by contradiction. Suppose $k_{\text{mon}} > k_{\text{duo}}$. Consider a symmetric steady-state subgame equilibrium under the payment profile $\mathbf{P}' = ((p_{\text{mon}}, p_{\text{mon}}), (p_{\text{mon}}, p_{\text{mon}}))$. Denote this steady-state subgame equilibrium by Σ', and let k'_1 denote the rate of customers served by firm 1 in Σ'. Observe that in Σ' the aggregate cost incurred by customers is $p_{\text{mon}} + c(mp_{\text{mon}} - 2k'_1)$. If $2k'_1 > k_{\text{mon}}$, then the rate of served customers in the equilibrium induced by $\mathbf{P}'$ is higher than the that of the equilibrium induced by $\mathbf{P}$, which contradicts the definition of $\mathbf{P}$ (given by (H.2)). On the other hand, if $2k'_1 < k_{\text{mon}}$, then the aggregate cost incurred by customers in Σ' is lower than that of the monopoly equilibrium. Since firm 1 offers the same price in both of the equilibria, we therefore must have $2k'_1 > k_{\text{mon}}$, which again is a contradiction. Therefore, $k_{\text{mon}} < k_{\text{duo}}$.

Part (ii) $p_{\text{mon}} < p_{\text{duo}}$. By (C.2) and (H.3),

$$c'(mp_{\text{mon}} - k_{\text{mon}}) = c'(mp_{\text{duo}} - k_{\text{duo}}) = -1/m.$$

Hence, the fact that $k_{\text{mon}} < k_{\text{duo}}$ implies that $p_{\text{mon}} < p_{\text{duo}}$ holds as well.

I (Re-)Defining the firms' actions as allocation choices

A classic way to handle "failure to launch problems"[20] in the two-sided platforms literature is redefining the actions of the platforms as quantity choices. We can repeat the same exercise here by defining the firm f's action as choosing the parameter k_f (the rate of customers who join the firm). After the vector (k_1, k_2) is fixed, then each firm f chooses price and wage to maximize profit subject to serving precisely k_f customers. We can show that, given (k_1, k_2), there is a unique tuple (p_f, w_f) for each firm f that maximizes the firm's profit subject to choosing price and wage. (We emphasize that the choices of p_f, w_f would not depend of the choices of p_{-f}, w_{-f}, so long as firm $-f$ serves k_{-f} customers.)

After redefining the firms' choices as above, we can compare the monopoly and duopoly equilibria exactly as in our main setup. The result would be the identical: when the labor pool is not sufficiently large, the firm chooses to serve a smaller rate of customers under the duopoly equilibrium (and with a higher price) compared to the monopoly equilibrium. The same proof as the proof of Theorem 6.5 also implies that the customers' average welfare is lower under the duopoly equilibrium when the labor pool is not sufficiently large.

J Incorporating service delays: the case of ride-sharing markets

J.1 Proof of Theorem 7.2

The proof follows similar steps as the proof of Theorem 4.1. For completeness, we include the full proof here. To present the theorem, we first need to define the system of equations that characterize the monopoly equilibrium. This is done in the next proposition, which is a counterpart to Proposition A.3 that provides these conditions in our main setting.

[20] In two-sided markets, equilibria in which a firm ends up having no customers on one side, e.g., because no customer has joined on the side are referred to as failure to launch problems.

Proposition J.1. *Any non-binding profit-maximizing equilibrium must satisfy*

$$\begin{cases} k = 1 - G(p + c(i)), \\ i = F(wm) - k \cdot (t(i) + 1), \\ k = -k_p(p, w) \cdot (p - w), \\ c'(i) = -\frac{kt'(i)+1}{mF'(w)}. \end{cases} \tag{J.1}$$

where p, w, k respectively denote the equilibrium price, wage, and the rate of customer requesting service.

Proof. The first and third equations are the same as those in Proposition A.3. The second equation writes the mass of idle workers as the mass of viable workers minus the sum of the mass of workers that are currently on trip to pick up a customer plus the mass of those who are driving the customer to her destination. The fourth equation is derived by equating the right-hand side of the third equation (i.e., the firm's first-order condition with respect to price) and the right-hand side of the equation $k = k_w(p, w) \cdot (p - w)$ (i.e., the firm's first-order condition with respect to wage). In file "pickup-time-Dp=Dw" we compute these right-hand sides and show that the fourth equation holds. $\square$

For the next, proposition, we recall that H, J respectively denote the inverse functions of the CDFs F, G.

Proposition J.2. *Let i, k, λ respectively denote the mass of idle workers, the rate of customers joining the firm, and the mass of viable workers at a profit-maximizing monopoly equilibrium where m is the size of the labor pool. Then, the i, k, λ must satisfy the following system of equations, denoted by $S(m)$:*

$$\lambda - (k \cdot (t(i) + 1) + i) = 0, \tag{J.2}$$

$$c'(i) + \frac{(kt'(i) + 1)\, H'\left(\frac{\lambda}{m}\right)}{m} = 0, \tag{J.3}$$

$$k \cdot (c'(i) - J'(1 - k)) - c(i) - H\left(\frac{\lambda}{m}\right) + J(1 - k) = 0, \tag{J.4}$$

Proof. (J.2) is just the same as the second equation in (J.1), written in terms of λ, i, k. (J.3) is the same as the fourth equation in (J.1), written in terms of λ, i, k. (J.4) is firm's first-order condition with respect to the choice of k (when the firm's decision problem is redefined as choosing k, λ). This equation is derived in an identical way as (A.3). $\square$

Let the functions $X(m,\lambda,i,k)$, $Y(m,\lambda,i,k)$, and $Z(m,\lambda,i,k)$ denote the left-hand sides of the equations (J.2),(J.3), and (J.4), respectively.

Following the proof of Theorem 4.1, to simplify the rest of the analysis, we extend the domains of the functions c, t, F, G so that: (i) their domains contain an interval $(-\epsilon, 0)$ for a $\epsilon > 0$, and (ii) c remains in $\mathbf{C^4}$ and strictly convex in the extended domain, (iii) The functions F, G remain in $\mathbf{C^4}$ in the extended domain, and (iv) F' remains decreasing in the extended domain.

Claim J.3. *There exists an open interval $I = (m_1, m_2)$ containing m_0 such that $S(m)$ has a solution at any $m \in I$. Furthermore, there exist unique and continuously differentiable functions $p(m), k(m) : \mathbb{R}_+ \to \mathbb{R}$ such that $(\lambda(m), k(m), i(m))$ is a solution to $S(m)$, for any $m \in I$.*

Proof. The proof is based on the Implicit Function Theorem. First of all, see that $(\lambda, k, i) = (0,0,0)$ is a solution to $S(m_0)$, i.e. $X(m_0,0,0,0) = Y(m_0,0,0,0) = Z(m_0,0,0,0) = 0$, where recall that the functions X, Y, Z denote the left-hand sides of the equations defining $S(m)$.

To apply the implicit function theorem, we need to prove that the Jacobian

$$J(m,k,p) = \begin{pmatrix} \frac{\partial X(m,\lambda,k,i)}{\partial \lambda} & \frac{\partial X(m,\lambda,k,i)}{\partial i} & \frac{\partial X(m,\lambda,k,i)}{\partial k} \\ \frac{\partial Y(m,\lambda,k,i)}{\partial \lambda} & \frac{\partial Y(m,\lambda,k,i)}{\partial i} & \frac{\partial Y(m,\lambda,k,i)}{\partial k} \\ \frac{\partial Z(m,\lambda,k,i)}{\partial \lambda} & \frac{\partial Z(m,\lambda,k,i)}{\partial i} & \frac{\partial Z(m,\lambda,k,i)}{\partial k} \end{pmatrix} \tag{J.5}$$

is invertible at point $(m_0 \lambda, i, k) = (m_0, 0, 0, 0)$. This is done in file "pickup-time-Jacobian". There, we show that the determinant of the above Jacobian is

$$\frac{(t(0)+1)\left(-m_0 c''(0)H'(0) - \frac{H'(0)H''(0)}{m_0}\right) + c''(0)m_0^2\left(-\frac{H'(0)}{m_0} - 2J'(0)\right) + H''(0)\left(-\frac{H'(0)}{m_0} - 2J'(0)\right)}{m_0^2}.$$

Since

$$\begin{aligned} &H'(0), J'(0) > 0, H''(0) \geq 0, \\ &c'(0) < 0, c''(0) > 0, \end{aligned}$$

the determinant is strictly negative. The Implicit Function Theorem therefore applies, and there exist an open interval $I \ni m_0$ and unique and continuously differentiable functions $\lambda(m), i(m), k(m)$ that solve the system $S(m)$ for all $m \in I$. $\square$

For the rest of the proof, we define $m_0 = \frac{1}{-c'(0)F'(0)}$.

Claim J.4. *Define $\lambda(m) = m \cdot F(w(m))$. Then, the following relations hold:*

$$\lim_{m \to m_0} k'(m) = 0, \quad \lim_{m \to m_0} k''(m) > 0$$
$$\lim_{m \to m_0} \lambda'(m) > 0.$$

Furthermore, the limits

$$\lim_{m \to m_0} k'''(m), \lim_{m \to m_0} \lambda''(m), \lim_{m \to m_0} \lambda'''(m)$$

exist and are finite.

Proof. This is a counterpart to Claim B.4 but for when there are service delays. The proof follows the same approach as the proof of Claim B.4. The functions $\lambda(m), i(m), k(m)$ are continuously differentiable functions, as shown in Claim J.3, by the Implicit Function Theorem. We use this fact to compute the limit of these functions and their (higher order) derivatives as m approaches m_0. In file "pickup-time-derivatives", we use implicit differentiation with respect to m from the system $S(m)$ to compute the closed-form expressions for the derivatives and their limits. We explain these derivations step by step in file "pickup-time-derivatives". □

Corollary J.5. *There exists $m_3 > m_0$ such that $\lambda'(m) \neq 0$ for any $m \in (m_0, m_3)$.*

Proof. This is a consequence of Claim J.4 and continuity of $\lambda'(m)$. □

Lemma J.6. *There exists a throughput-maximizing equilibrium at m iff $m > m_0$.*

Proof. The proof uses the proof of Lemma B.1 from our original setup (where the delay times $t(i)$ are equal to 0 for all i). First, we show that any m that is not feasible in the original setup cannot be feasible here in the new setup in which the delay times can be positive. (Recall the definitions of the feasibility notion from the proof of Lemma B.1.) To see this, fix m, and observe that any payment profile (p, p) that is not feasible at m in the original setup is also not feasible at m here in the new setup. The reason simply is that there would be fewer idle workers available under any payment profile in the new setup than in the original setup, because of the positive delay times. Therefore, the waiting cost will remain at least as large as 1 in the new setup as well, which means (p, p) is not feasible at any $m \leq m_0$ in the new setup. Hence, to complete the proof, it remains to show that any $m > m_0$ is feasible in the new setup.

To this end, we show that if $m = m_0 + \delta$ for some $\delta > 0$, then m is feasible. To prove this, it suffices to show this for an arbitrary small $\delta > 0$, because if m is feasible, then so would be any $\tilde{m} > m$. Recall that, by Claim J.3, there exists an open interval I around m_0 such that $S(m)$ has a unique solution in that interval, given by $(\lambda(m), i(m), k(m))$. Furthermore, recall that by Claim J.4, $k'(m_0) > 0$. The continuity of $k'(m)$ in the interval I, which is guaranteed by Claim J.3, then implies that $k(m) > 0$ for all $\{m \in I : m \geq m_0\}$. Consequently, there exists $\bar{\delta} > 0$ such that, for any positive $\delta < \bar{\delta}$, $m = m_0 + \delta$ is feasible. $\square$

To prove the theorem, we will show that there exists $\hat{m} > m_0$ such that $e'(m) > 0$, $w'(m) > 0$, and $(u^W)'(m)$ hold for all $m \in (m_0, \hat{m})$. This will be done in Proposition J.7, Proposition J.10, and Proposition J.11.

Proposition J.7. *There exists a threshold $\hat{m}$ such that for all $m \in (m_0, \hat{m})$, $e'(m) > 0$.*

Proof. First, we prove the following claims.

Claim J.8. *There exists $m_4 > m_0$ such that $e'(m)$ exists at all $m \in (m_0, m_1)$.*

Proof. First, observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Because $\lambda(m) > 0$ for all $m > m_0$, and because $k(m), k'(m)$ and $\lambda'(m)$ exist and are finite for m sufficiently close to m_0 (by Claim J.4), $e'(m)$ exists and is finite for m sufficiently close to m_0.

$\square$

Claim J.9. $\lim_{m \to m_0} e(m) = 0$.

Proof. First, observe that

$$\lim_{m \to m_0} e(m) = \lim_{m \to m_0} \frac{k(m)}{\lambda(m)} = \lim_{m \to m_0} \frac{k'(m)}{\lambda'(m)}.$$

L'Hôpital's Rule is applicable here by Corollary J.5. In file "pickup-time-derivatives" we compute the closed-form expression for the right-hand side, and thereby prove the claim. $\square$

Continuity of $e(m)$ at m_0 is ensured by Claim J.9. The rest of the proof is as follows. We will show that $\lim_{m \to m_0} e'(m)$ exists and is positive. This would imply that $e'(m_0)$ must also exist, and in fact that

$$e'(m_0) = \lim_{m\to m_0} e'(m).$$

(This is a consequence of L'Hôpital's Rule. See, for example, [Wikipedia 2017] for a proof.) Once we have shown this, the proof is complete: because $e'(m_0) > 0$, then there must exists $\hat{m}$ such that $e'(m) > 0$ for $m \in [m_0, \hat{m}]$, i.e. $e(m)$ is an increasing function over $[m_0, \hat{m}]$.

First, observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Therefore,

$$\lim_{m\to m_0} e'(m) = \lim_{m\to m_0} \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

The L'Hôpital's Rule is applicable because $\lim_{m\to m_0} k(m) = k(m_0) = 0$ and $\lim_{m\to m_0} \lambda(m) = \lambda(m_0) = 0$ hold, as we observed in Claim J.3 and its proof. Thereby, we can write

$$\begin{aligned}\lim_{m\to m_0} e'(m) &= \lim_{m\to m_0} \frac{k''(m)\lambda(m) - \lambda''(m)k(m)}{2\lambda(m)\lambda'(m)} \\ &= \lim_{m\to m_0} \frac{k'''(m)\lambda(m) + k''(m)\lambda'(m) - \lambda'''(m)k(m) - \lambda''(m)k'(m)}{2\lambda'(m)^2 + 2\lambda(m)\lambda''(m)}, \end{aligned} \tag{J.6}$$

where (J.6) is due to a second application of L'Hôpital's Rule. We can simplify this equality further and write

$$\lim_{m\to m_0} e'(m) = \lim_{m\to m_0} \frac{k''(m)\lambda'(m)}{2\lambda'(m)^2} = \lim_{m\to m_0} \frac{k''(m)}{2\lambda'(m)} > 0, \tag{J.7}$$

where the above relation holds by Claim J.4. This proves the promised claim. □

Proposition J.10. *There exists $\hat{m} > m_0$ such that for all $m \in [m_0, \hat{m})$, $w'(m) > 0$.*

Proof. To prove the claim, we first note that $\lambda'(m) = F(w(m)) + mw'(m)F'(w(m))$. Hence, to prove that $\lim_{m\to m_0} w'(m) > 0$, it would suffice to show that $\lim_{m\to m_0} \lambda'(m) > 0$. This was proved by Claim J.4. Continuity of $w'(m)$ then implies that there exists $\delta > 0$ such that $w'(m) > 0$ for $m \in [m_0, m_0 + \delta]$. This proves the claim. □

Proposition J.11. *There exists a threshold $\hat{m} > m_0$ such that for all $m < \hat{m}$, $(u^W)'(m) > 0$.*

Proof. The proof is similar to the proof of Proposition B.10. □

K Workers accepting offers lose outside option permanently

Throughout this section, we suppose that c is a regular cost function and that F, G are the uniform distribution. The distributional assumption is made for analytical tractability.

K.1 Throughput maximizing monopoly equilibrium

Consider a steady-state monopoly equilibrium where the price, wage, and the rate of customers joining the firm are respectively given by p, w, k. We have $p = w$, since the steady-state equilibrium is throughput maximizing.

Let $v(r)$ denote the steady-state payoff a worker with outside option r who joins the firm. Then, $v(r) = wk/\lambda$, where λ denotes the total mass of viable workers. Let s denote the supremum of the outside options of the set of viable workers at the equilibrium. Since a worker with outside option s is indifferent between joining and not joining the firm, then $s = v(s) = wk/\lambda$. On the other hand, since all workers with an outside option lower than s join the firm, then $\lambda = ms$. The two latter equations imply that $w = \lambda^2/(mk)$. Since $w = p$, we can then write the rate of customers joining the firm as

$$k = 1 - c(\lambda - k) - \frac{\lambda^2}{mk} \tag{K.1}$$

This is the market-clearing condition. At a throughput maximizing equilibrium, the firm's objective can be formulated as choosing λ, k such that k is maximized subject to satisfying the above equation. We then can write the first-order conditions with respect to k, λ by implicit differentiation from the above equation. That gives:

$$\begin{cases} c'(\lambda - k) + \frac{\lambda^2}{k^2 m} = 1, & \text{The first-order condition for } k \\ c'(\lambda - k) + \frac{2\lambda}{km} = 0. & \text{The first-order condition for } \lambda \end{cases} \tag{K.2}$$

The above equations imply that

$$c'(\lambda - k) = -\frac{2\lambda}{km} = \frac{k^2 m - \lambda^2}{k^2 m} \tag{K.3}$$

Solving for λ gives $\lambda = k - k\sqrt{m+1}$ or $\lambda = k\sqrt{m+1} + k$. The former root is negative and hence is ruled out. Therefore,

$$\lambda = (1 + \sqrt{m+1})k \tag{K.4}$$

Plugging the above solution into $w = \lambda^2/(mk)$ gives

$$w = \frac{\left(\sqrt{m+1}+1\right)^2 k}{m}. \tag{K.5}$$

Plugging the two latter equations into (K.1) gives the simplified market-clearing condition

$$c\left(k\sqrt{m+1}\right) = 1 - \frac{2k\left(m + \sqrt{m+1}+1\right)}{m} \tag{K.6}$$

We now can write the first-order condition for k by implicit differentiation from the above equation, which is:

$$-\sqrt{m+1}c'\left(k\sqrt{m+1}\right) - \frac{2\left(m+\sqrt{m+1}+1\right)}{m} = 0,$$

solving which gives

$$c'\left(k\sqrt{m+1}\right) = -\frac{2\left(m+\sqrt{m+1}+1\right)}{m\sqrt{m+1}}.$$

Therefore,

$$k = \frac{c'^{-1}\left(-\frac{2\left(m+\sqrt{m+1}+1\right)}{m\sqrt{m+1}}\right)}{\sqrt{m+1}} \tag{K.7}$$

gives the equilibrium value for k. With slight abuse of notation, we denote the right-hand side of the above equation by $k(m)$.

First, note that at any steady-state equilibrium, the constraints $\lambda > k$ and $k > 0$ must be non-binding: the former constraint must be non-binding because c is a regular cost function, and the latter constraint must be non-binding by the definition of a steady-state equilibrium. Hence, any solution the firm's problem must be an interior solution to (K.2). Therefore, in case of existence of an equilibrium, the equilibrium value of k must be given by (K.7).

First, we observe that $f(m) = -\frac{2\left(m+\sqrt{m+1}+1\right)}{m\sqrt{m+1}}$ is strictly increasing in m. Define $m_0 = \frac{4(1-c'(0))}{c'(0)^2}$, and observe that $f(m_0) = 0$. Hence, $f(m) < 0$ iff $m < m_0$. This also implies that $k(m) < 0$ iff $m < m_0$, and $k(m) = 0$ iff $m = m_0$. Therefore, no monopoly equilibrium can exist when $m \leq m_0$, i.e. no $m \leq m_0$ is feasible.

Define $\lambda(m) = (1+\sqrt{m+1})k(m)$ according to (K.4). Observe that for all $m > m_0$, $k(m) > 0$, and moreover, $\lambda(m) > k(m)$ holds by (K.4). Hence, $k(m), \lambda(m)$ give an interior solution to the firm's first-order conditions given by (K.2), and induce a steady-state equilibrium. By the above analysis, we have the following lemma. Let $s(m)$ denote the outside option of the worker with the largest outside option who joins the firm at the monopoly equilibrium, given that the labor pool has size m.

Lemma K.1. *Let $\underline{m} = \frac{4(1-c'(0))}{c'(0)^2}$. When the firm's objective is throughput maximization, no monopoly equilibrium exists at m if $m \leq \underline{m}$, Furthermore, at any $m > \underline{m}$ there exists a unique monopoly throughput maximizing equilibrium characterized by the following parameters*

$$
\begin{aligned}
k(m) &= \frac{c'^{-1}\left(-\frac{2\left(m+\sqrt{m+1}+1\right)}{m\sqrt{m+1}}\right)}{\sqrt{m+1}}, \\
\lambda(m) &= (1+\sqrt{m+1})k(m), \\
p(m) = w(m) &= \frac{\left(\sqrt{m+1}+1\right)^2 k(m)}{m}, \\
s(m) &= \frac{w(m)k(m)}{\lambda(m)}.
\end{aligned}
$$

Before presenting the theorem, we define $u^W(m) = \frac{1}{r(m)} \cdot \int_0^{r(m)} w(m)e(m)\,\mathrm{d}\,x$ to be the average welfare of workers who join the firm. We recall that $e(m) = \frac{k(m)}{\lambda(m)}$.

Theorem K.2. *When the firm's objective is throughput maximization, there exists $\underline{m}$ such that a monopoly equilibrium exists at m iff $m > \underline{m}$, and there exists $\hat{m} > \underline{m}$ such that for all $m \in (\underline{m}, \hat{m})$ $w'(m)$, and $(u^W)'(m)$ are positive, whereas $e'(m)$ is negative.*

Proof. We compute the closed-form expression for $w'(m_0)$ in file "outsideoption-w" as

$$
w'(m_0) = -\frac{\left(c'(0)-1\right)c'(0)^4(c'^{-1})'\left(c'(0)\right)}{4\left(c'(0)-2\right)^2}.
$$

Observe that $(c')^{-1}$ is an increasing function, which shows that $w'(m_0) > 0$. Also, observe that the $u^W(m) = w(m)e(m)$. In file "outsideoption-w", we compute

$$
(u^W)'(m_0) = -\frac{c'(0)^5(c'^{-1})'(0)}{8\left(c'(0)-2\right)^2}.
$$

The right-hand side is positive because $(c')^{-1}$ is an increasing function.

It remains to prove the claim about $e'(m)$. Observe that by Lemma K.1, $e(m) = \frac{1}{1+\sqrt{m+1}}$, which is a decreasing function in m. $\square$

K.2 Profit-maximizing monopoly equilibrium

We start by formulating the firm's problem by writing it in terms of the choice variables k, λ. To this end, we recall from the previous section the equation $w = \frac{\lambda^2}{mk}$, which holds here as well, by the same proof. Also, observe that by the market-clearing condition, $p = 1 - k - c(\lambda - k)$. Therefore, we can write the firm's profit as

$$(p - w)k = \left(1 - k - c(\lambda - k) - \frac{\lambda^2}{mk}\right) k.$$

Let the function $\pi(k, \lambda)$ denote the right-hand side of the above equation. The firm's objective can be formulated as choosing k, λ such that $\pi(k, \lambda)$ is maximized subject to $\lambda > k > 0$. We then can write the first-order conditions with respect to k, λ by implicit differentiation from the above equation. Simplifying the first-order conditions $\frac{\partial \pi(k,\lambda)}{\partial k} = 0$ and $\frac{\partial \pi(k,\lambda)}{\partial \lambda} = 0$ gives the following system of equations

$$\begin{cases} kc'(\lambda - k) + 1 = c(\lambda - k) + 2k, & \text{The first-order condition for } k \\ kc'(\lambda - k) + \frac{2\lambda}{m} = 0. & \text{The first-order condition for } \lambda \end{cases} \tag{K.8}$$

First, note that at any steady-state equilibrium, the constraints $\lambda > k$ and $k > 0$ must be non-binding: the former constraint must be non-binding because c is a regular cost function, and the latter constraint must be non-binding by the definition of a steady-state equilibrium. Hence, any solution the firm's problem must be an interior solution to (K.8).

Through out the rest of this section, for analytical tractability, we focus on the case when c is an exponential function, i.e. $c(x) = e^{-\gamma x}$ where $\gamma > 0$ is a constant. Then, $c'(x) = -\gamma c(x)$. This equation together with (K.8) imply that

$$\lambda = \frac{km(\gamma - 2\gamma k)}{2 + 2\gamma k}. \tag{K.9}$$

Plugging this in the firm's profit function implies that the firm's profit at the profit-maximizing equilibrium is given by

$$\left(1-k-c\left(-\frac{k(2\gamma k(m+1)-\gamma m+2)}{2\gamma k+2}\right)-\frac{\gamma^2 k(1-2k)^2 m}{4(\gamma k+1)^2}\right)\cdot k.$$

Let us denote the above expression by $\Pi(k)$. Since any profit-maximizing equilibrium is non-binding, then the equilibrium value of k at a profit-maximizing equilibrium belongs to $\arg\max_{k\in(0,1)}\Pi(k)$. After fixing $\gamma>0$, we solve this optimization problem numerically to compute the equilibrium value of k at different values of m. The equation $w=\frac{\lambda^2}{mk}$ then allows us to also compute the equilibrium wage at different values of m. This computation is done in file "plot-outsideoption". The resulting plot is presented in Figure 7 (left panel). We remark that from Lemma K.1 it implies that a profit-maximizing monopoly equilibrium exists iff $m>\underline{m}$, where $\underline{m}=\frac{4(1-c'(0))}{c'(0)^2}$.

Figure 13 similarly compares the workers' average welfare under Assumption 7.1 (left panel) and under the assumption in our main setup (right panel). We note that, under Assumption 7.1, the workers' average welfare at a monopoly equilibrium is defined by $\frac{w^*k^*}{\lambda^*}$ (where k^*,λ^*,w^* denote the equilibrium levels of these parameters), whereas it is defined by (4.2) in our main setup, i.e., $\left(w^*+\frac{k^*}{m}\right)/2$.

Similar to the case of wage, we observe that the complementarity effects between workers are magnified under Assumption 7.1, in the sense that average welfare remains increasing in m over a larger interval in the left panel. The intuition remains similar to the case of wage, as discussed in Section 7.1. This plot is created in file "plot-outsideoption-avew".

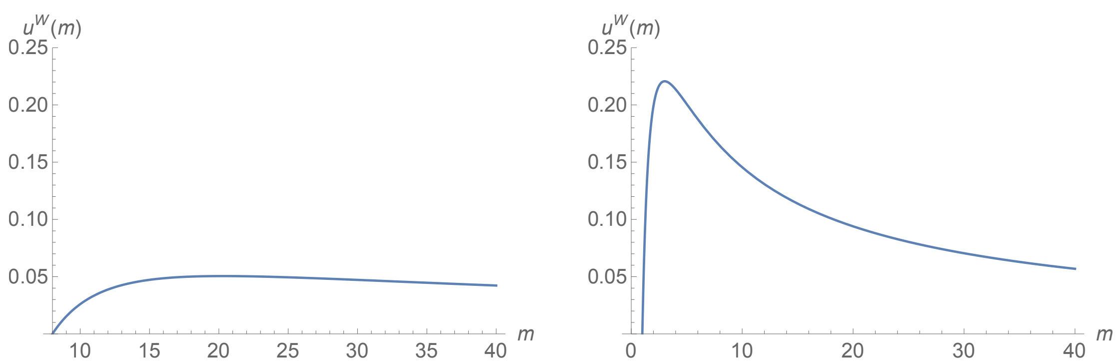

Figure 13: Equilibrium level of workers' average welfare as functions of m for when F,G are the uniform distribution over $[0,1]$ and $c(i)=e^{-i}$. The left and right plots respectively correspond to the alternative and original assumptions.

Figure 14 plots the average employment time under Assumption 7.1, which is decreasing in m. We recall that in our main setup, however, the average employment time is increasing in m when m is not sufficiently large. This plot is created in file "plot-outsideoption-e".

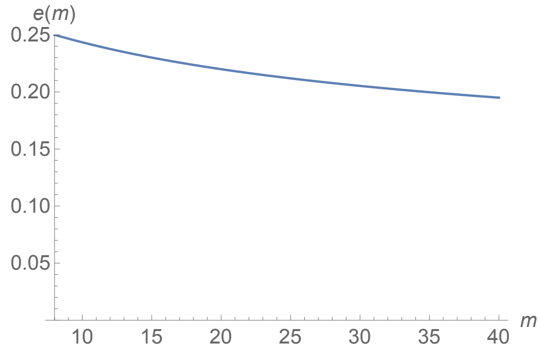

Figure 14: Equilibrium level of workers' average employment time as functions of m for when F, G are the uniform distribution over $[0, 1]$ and $c(i) = e^{-i}$.

L The role of convexity

To convey the intuition for the case of non-convex cost functions, we first consider an example where c is an affine function and F, G are the uniform distribution. Then, in Section L.1 we consider general concave cost functions.

In this example, $c(i) = \max\{0, 1-i\}$. A mass m is feasible iff there exists $p > 0$ such that $p + c(mp) < 1$. Hence, a monopoly equilibrium exists iff $m > 1$. Let $m_0 = 1$. Fix an arbitrary $m > m_0$. We investigate whether there exists a beneficial deviation $p' = p(m) + \epsilon_p$ and $w' = w(m) + \epsilon_w$ which results in the same level of customers, but increases profit. Suppose we increase wage by ϵ_w; this results in a decrease of $m\epsilon_w$ in waiting cost. Therefore, we can set $\epsilon_p = m\epsilon_w$. This results in the same rate of customers joining the firm, while increasesing the profit per service. So, at the monopoly equilibrium, such a deviation should not be possible. That is, we must have $i(m) = 1$. In other words, when $\gamma = 1$, the monopolist always maintains the same level of idle workers, $i = 1$. The monopoly equilibrium then must be the solution to

$$\begin{aligned} &\max_{p,w\geq 0} \Pi(p, w) \\ \text{s.t.}\ \ & k^* = 1 - p^*, \\ & mw^* - k^* = 1. \end{aligned}$$

Solving by setting the first-order condition equal to 0 implies that $w(m) = \frac{1+3m}{2m+2m^2}$, which is decreasing in m. The interpretation is simple: Let m marginally increase and ignore the changes in $k(m)$. If the level of wage stays the same, the number of idle workers will be larger than 1. The monopolist therefore decreases wage as m goes up.

L.1 Concave cost functions

We repeat the above exercise for more general cost functions and show that when c is concave or affine, equilibrium wage decreases with m. This is done by showing that in such cases, $c(i^*) = 0$ holds at any monopoly equilibrium. Given this fact, we can repeat the above exercise identically, which would imply that wage decreases with m.

Lemma L.1. *There is at most one monopoly equilibrium* (p^*, w^*, k^*) *that satisfies* $c(i^*) = 0$.

Proof. At a monopoly equilibrium, $c(i^*) = 0$ implies that $i^* = 1$. Therefore, $i^* = mw^* - k^*$ implies that $w^* = \frac{1+k^*}{m}$. This equation, together with the equation $k^* = 1 - p^* - c(i^*)$ and the first-order condition form a system of equations that characterize the equilibrium. This system has a unique solution, as shown in file "c0-eqm". □

Lemma L.2. *Suppose that* c *is strictly concave. Then, the (unique) monopoly equilibrium* (p^*, w^*, k^*) *satisfies* $c(i^*) = 0$.

Proof. We prove that $c(i^*) = 0$ must be satisfied at any equilibrium. The proof of uniqueness then follows from (L.1). Proof by contradiction. We consider the deviation that increases both price and wage by $\epsilon > 0$, i.e. $p^\# = p^* + \epsilon$ and $w^\# = w^* + \epsilon$, and prove profit is decreasing along this direction. Define the function

$$\Pi_\epsilon(p, w) \equiv k(p + \epsilon, w + \epsilon) \cdot (p - w).$$

Observe that

$$\frac{\mathrm{d}\,\Pi_\epsilon(p^*, w^*)}{\mathrm{d}\,\epsilon} = \frac{\mathrm{d}\,k(p + \epsilon, w + \epsilon)}{\mathrm{d}\,\epsilon} \cdot (p - w), \tag{L.1}$$

$$\frac{\mathrm{d}^2\,\Pi_\epsilon(p^*, w^*)}{\mathrm{d}\,\epsilon^2} = \frac{\mathrm{d}^2\,k(p + \epsilon, w + \epsilon)}{\mathrm{d}\,\epsilon^2} \cdot (p - w). \tag{L.2}$$

Next, in file "concave-c" we compute

$$\frac{\mathrm{d}\,k(p^* + \epsilon, w^* + \epsilon)}{\mathrm{d}\,\epsilon} = 0$$

$$\frac{\mathrm{d}^2\,k(p^* + \epsilon, w^* + \epsilon)}{\mathrm{d}\,\epsilon^2} = -\frac{m(m+1)c''(mw^* - k)}{\left(c'(mw^* - k) - 1\right)^2} > 0, \tag{L.3}$$

where the first inequality holds because the rate of customers who join, k, must not increase by deviation (ϵ, ϵ) and the second inequality holds because $c''(i) < 0$ for all $i \geq 0$. Now, (L.2) and (L.3) together imply that the deviation (ϵ, ϵ) increases the profit, which is a contradiction. $\square$

Lemma L.3. *Suppose that c is affine. Then, the (unique) monopoly equilibrium (p^*, w^*, k^*) satisfies $c(i^*) = 0$.*

Proof. We prove that $c(i^*) = 0$ must be satisfied at any equilibrium. The proof of uniqueness then follows from (L.1). Consider an arbitrary monopoly equilibrium, namely (p, w, k) such that $mw - k > 0$. Also, consider an $\epsilon > 0$ sufficiently small. We show that the deviation that increases both price and wage by $\epsilon > 0$, i.e. $p^\# = p^* + \epsilon$ and $w^\# = w^* + \epsilon$ does not change the profit. The proof is by contradiction. Suppose it does. If the deviation increases the profit, then we reach a contradiction because we supposed (p, w, k) is a monopoly equilibrium. If the deviation $(+\epsilon, +\epsilon)$ decreases the profit, then the deviation $(-\epsilon, -\epsilon)$ must increase the profit. (This is a straight-forward consequence of affinity of c.) Therefore, the deviation (ϵ, ϵ) must not change the profit. This implies that, without changing the profit, we can change p^*, w^* by moving along the direction $(-\epsilon, -\epsilon)$ until the number of idle workers is equal to 0. But at this point, the rate of customers who join, and thereby the profit, should be qual to 0. Contradiction. $\square$